2019 Four-by-Four Competition
Solutions

1. **If 5 Credenzas are equivalent to 7 Dalmatians and 4 Dalmatians are equivalent to 3 Eggs, how many Eggs are equivalent to 600 Credenzas?**

   600 Credenzas is 120 times 5 Credenzas, so it is equivalent to $120 \cdot 7$ Dalmatians. That is $30 \cdot 7$ times 4 Dalmatians, so it is equivalent to $30 \cdot 7 \cdot 3 = 630$ Eggs.

2. **When 89 people were surveyed, 47 said they liked Vanilla, 64 said they liked Chocolate, and 34 said they liked both. How many said they liked neither Vanilla nor Chocolate?**

   If 34 liked both, $47 - 34 = 13$ liked Vanilla but not Chocolate, so that $64 + 13 = 77$ liked at least one, so that $89 - 77 = 12$ liked neither.

3. **What is the smallest positive integer whose leftmost two digits can be read as a two-digit number which is the square of its rightmost digit?**

   A rightmost digit of 4 gives leftmost digits of 16. 5 gives 25, 6 gives 36, 7 gives 49, 8 gives 64, and 9 gives 81. You might think the smallest answer is thus 416, but it turns out that it's 25 (rather than 255). Because of the equal digits, the leftmost two digits include the rightmost digit.

4. **What is the largest possible volume of a right rectangular prism with integer side lengths and a surface area of 100 m^2?**

   The surface area is $2(lw + lh + wh) = 100$, so $lw + lh + wh = 50$. For a maximum volume, we'd prefer the dimensions to be equal, so each part of that sum would be ~ 16. Could two dimensions be 4? If so, we'd have $16 + 4h + 4h = 50$, so that $16 + 8h = 50$, and $8h = 34$, which is sadly impossible. What about 5 & 3? That would give $15 + 8h = 50$, which also doesn't work. 6 & 2 would give $12 + 8h = 50$, which still doesn't work. How about 7 & 2? It gives $14 + 9h = 50$, so that $9h = 36$ and $h = 4$, for an answer of $7 \cdot 2 \cdot 4 = 56$.

5. **If today is Thursday, what day of the week was it 823 days ago?**

   700 days ago was Thursday, leaving 123 more days. 140 days before that was Thursday, but now we're 17 days too early. 14 days later would be Thursday, and three days after that would be Friday, Saturday, Sunday.

6. **Evaluate:** $\cfrac{2}{5+\cfrac{2}{5+\cfrac{2}{5+\cdots}}}$

   We can write $x = \frac{2}{5+x}$, which becomes $x^2 + 5x = 2$, then $x^2 + 5x - 2 = 0$. The quadratic formula gives $x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1(-2)}}{2 \cdot 1} = \frac{-5 \pm \sqrt{25+8}}{2} = \frac{-5 \pm \sqrt{33}}{2}$. Of these two values, only the positive one is plausible, for an answer of $\frac{\sqrt{33}-5}{2}$.

7. **Express the base 12 numeral $5A9B_{12}$ as a base 10 numeral.**

   From right to left, the digits in base 12 represent $12^0 = 1$, $12^1 = 12$, $12^2 = 144$, etc. Base 12 uses the digits 0-9, plus A, which represents 10, and B for 11. $11 \cdot 1 + 9 \cdot 12 + 10 \cdot 144 + 5 \cdot 1728 = 11 + 108 + 1440 + 8640 = 10{,}199$

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8. What is the height, in meters, of a cylinder with a base radius of 9 m if the volume of the cylinder is twice the volume of a cone with a base radius of 6 m and a height of 6 m?

We can write $\pi \cdot 9^2 h = 2 \cdot \frac{1}{3}\pi \cdot 6^2 \cdot 6$, which becomes $81h = 144$, so that $h = \frac{144}{81} = \frac{16}{9}$.

9. Evaluate the value of the function $\mathbf{Arctan}(-1)$ in radians.

The Arctan function is defined to have a range of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We're looking for a value of θ that satisfies $\tan(\theta) = -1$, and the one in the given range is $-\frac{\pi}{4}$.

10. What are the coordinates, in the form (x, y), of the y-intercept of the line $8x - 7y = 112$?

The y-intercept is where $x = 0$, which gives $8 \cdot 0 - 7y = 112$, giving $-7y = 112$, and finally $y = -16$, for an answer of $(0, -16)$.

11. What is the equation (in the form $Ax + By + Cz = D$ where A is positive and A, B, C, and D are collectively relatively prime) of the plane through the point $(9, -8, -3)$ and perpendicular to the vector $\langle -9, 4, 5 \rangle$?

The equation will be of the form $9x - 4y - 5z = D$. Substituting gives $9 \cdot 9 - 4(-8) - 5(-3) = 81 + 32 + 15 = 128$, for an answer of $9x - 4y - 5z = 128$.

12. What is the largest real value of p that can be part of a solution to $3p^2 + 8pq + 6q^2 = 216$?

This can be viewed as a quadratic in terms of either p or q, with the other one being part of the constants. For the largest value of p, q will have exactly one value, so the discriminant of its quadratic equation should be 0. $(8p)^2 - 4 \cdot 6 \cdot (3p^2 - 216) = 0$ becomes $64p^2 - 72p^2 + 5184 = 0$, then $8p^2 = 5184$ and $p^2 = 648$, so $p = \sqrt{648} = 2\sqrt{162} = 6\sqrt{18} = 18\sqrt{2}$.

13. What is the distance between the points with coordinates $(4, 3)$ and $(-8, 5)$?

The Distance Formula gives $\sqrt{\left(4 - (-8)\right)^2 + (3 - 5)^2} = \sqrt{12^2 + 2^2} = 2\sqrt{6^2 + 1^1} = 2\sqrt{37}$.

14. What is the largest palindrome less than 130987 which is a multiple of 3 but contains no digits which are positive multiples of 3?

130031 is the largest palindrome, but it has a multiple of 3, so we should go down to 129921, but this also has multiples of 3. Dropping down to 128821, there are no multiples of 3, but the entire number is not a multiple of 3. 126621 would be, but it's bad, as is 123321, so we end up at 120021.

15. Arrange the numbers below in increasing order (e.g. ABCD).

A. $\frac{5}{2}$ **B.** $\sqrt{6}$ **C.** $\frac{3\pi}{4}$ **D.** $\frac{44}{19}$

I think it's easiest to think in terms of decimals for problems like this. $A = 2.5$, $B \approx 2.45$, $C \approx \frac{3 \cdot 3.14}{4} = \frac{9.42}{4} = 2.355$, and $D \approx 2.31$, for an answer of $DCBA$.

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16. What is the sum of the even integers between 83 and 289?

We're adding up the evens from 84 to 288, inclusive. Outer pairs sum to $84 + 288 = 372$, and there are $\frac{288-84}{2} + 1 = \frac{204}{2} + 1 = 102 + 1 = 103$ numbers, for a total of $\frac{103 \cdot 372}{2} = 103 \cdot 186 = 19{,}158$.

17. What is the area, in square meters, of a triangle with sides measuring 7 m, 8 m, and 5 m?

Stewart's Theorem gives $\sqrt{10 \cdot 5 \cdot 3 \cdot 2} = 10\sqrt{3}$.

18. Evaluate in terms of i $(= \sqrt{-1})$: $(1 + 2i)(i)^{34}(5 - i) + (2i)^3$

$$(1 + 2i)(i)^{34}(5 - i) + (2i)^3 = (1 + 2i)(5 - i)(-1) - 8i = -5 - 2 - 10i + i - 8i$$
$$= -7 - 17i$$

19. Evaluate: $869946 \div 147$

The standard algorithm gives 5918.

20. What is the largest five-digit integer that can be "broken" into four two-digit numbers that are pairwise relatively prime? E.g. 12,345 can be "broken" into 12, 23, 34, and 45 (but not all pairs are relatively prime).

Starting with $99 = 3^2 \cdot 11$, we can have our second sub-number be $98 = 2 \cdot 7^2$, the third can be 89 (prime), and the fourth can be 97 (prime), for an answer of 99897

21. How many liters of a 40% acid solution should you add to 9 liters of a 70% acid solution to produce a 60% acid solution?

Our desired 60% solution is 20 away from 40% and 10 away from 70%, for a ratio of 2:1. Because of this, we should mix the initial solutions in the ratio 1:2, for an answer of $\frac{9}{2}$.

22. What is the missing term of the sequence 1, 1, 3, 5, 11, 21, 43, 85, 171, ___, …?

We originally intended this to be a recursive sequence where $a_n = 2a_{n-1} + a_{n-2}$, but the more apparent pattern turned out to be $a_n = 2a_{n-1} - 1$ alternating with $a_n = 2a_{n-1} + 1$. The latter way, the missing term is $2 \cdot 171 - 1 = 342 - 1 = 341$.

23. What is the perimeter, in meters, of an isosceles right triangle with an area of 3 m^2?

We can write $\frac{1}{2}s^2 = 3$, so $s^2 = 6$ and $s = \sqrt{6}$. This means that the hypotenuse is $\sqrt{6} \cdot \sqrt{2} = \sqrt{12} = 2\sqrt{3}$, for an answer of $2\sqrt{6} + 2\sqrt{3}$.

24. What is the sum of the 23 smallest positive perfect squares that are not also perfect fourth powers?

The squares that are also fourth powers are squares of squares, so the squares of 1, 4, 9, 16, 25, 36, 49, etc. To get the 23 numbers we want, we need the squares of 1-23, but that includes 1, 4, 9, and 16, so we need 1-27, but that includes 25, so we need 1-28. The sum of the first 28 squares is $\frac{28 \cdot 29 \cdot 57}{6} = 14 \cdot 29 \cdot 19 = 7714$, but that includes the squares of 1, 4, 9, 16, and 25, which we need to remove, so we'll subtract $1 + 16 + 81 + 256 + 625 = 979$, for an answer of $7714 - 979 = 6735$.

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25. What is the area, in square meters, of a parallelogram with sides measuring 4 m and 3 m and an internal angle measuring $150°$?

If one angle is 150, that means another is $180 - 150 = 30$, so a 30-60-90 triangle can be made so that the height of the parallelogram relative to its base of 3 is $\frac{4}{2} = 2$, for an answer of $3 \cdot 2 = 6$.

26. How many subsets of the set of counting numbers less than 10 contain at least two prime numbers?

For each of 1, 4, 6, 8, and 9, we can either have or not have them in a subset (2 choices), for a total of $2^5 = 32$ ways to include them. For 2, 3, 5, and 7, we want at least two. There is 1 way to have all 4, 4 ways to have 3, and 6 ways to have 2, for a total of $1 + 4 + 6 = 11$ ways to include them and a final answer of $11 \cdot 32 = 352$.

27. What are the coordinates, in the form (x, y), of the point of intersection of the lines $4x + 3y = 24$ and $y = -2x + 4$?

The latter can be $2x + y = 4$, which can be doubled and subtracted from the former to give $y = 16$, so that $x = -6$, for an answer of $(-6, 16)$.

28. Arithmetic sequence Z has $z_1 = 31$ and $z_2 = 967$, while arithmetic sequence Y has $y_1 = 613$ and $y_2 = 2329$. What is the smallest positive difference between a term in sequence Z and a term in sequence Y?

The terms of sequence Z differ by multiples of $967 - 31 = 936$, and the terms of sequence Y differ by multiples of $2329 - 613 = 1716$. Because $936 = 3^2 \cdot 104 = 3^2 \cdot 2^3 \cdot 13$ and $1716 = 2^2 \cdot 429 = 2^2 \cdot 3 \cdot 143 = 2^2 \cdot 3 \cdot 11 \cdot 13$, the GCF of 936 and 1716 is $2^2 \cdot 3 \cdot 13 = 156$, so the differences between terms of the two sequences can vary by multiples of 156. The difference between the initial terms is $613 - 31 = 582$, which can be chiseled down to $582 - 156 = 426$, then $426 - 156 = 270$, then $270 - 156 = 114$, and finally $114 - 156 = -42$, for an answer of 42.

29. What is the name for the locus of points satisfying $9x^2 + 5xy - 4y^2 - 9x + 3y = 298$?

The presence of an xy term indicates that whatever this conic is, it's at an angle to the coordinate axes. There are ways of dealing with this, but if the only thing you care about is the shape, you don't have to. Our old friend the discriminant returns! In this case, $B^2 - 4AC = 5^2 - 4 \cdot 9(-4) = 25 +$ we-don't-care-what, because it will be positive, which means the graph will be a hyperbola. Zero would have meant a parabola, and negative would have indicated an ellipse.

30. If it is currently 2:18:58 PM, what time will it be in 42398 seconds? Include AM or PM in your answer.

42398 seconds is $42398 \div 60 = 706$ minutes and 38 seconds, which is $706 \div 60 = 11$ hours and 46 minutes (and 38 seconds). Thus, the new time will be 46:38 after 1:18:58 AM, which is 38 seconds after 2:04:58 AM, for an answer of 2:05:36 AM.

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31. What is the largest possible area of a triangle with vertices at the vertex, the y-intercept, and an x-intercept of the parabola with equation $y = 3(x+1)^2 - 12$?

The vertex is at $(-1,-12)$, the y-intercept is at $(0,-9)$, and the x-intercepts are $(1,0)$ and $(-3,0)$. The latter x-intercept is farther from the line through the vertex and y-intercept, so will make the larger area, which we can find as a rectangle minus some triangles. $A = 12 \cdot 3 - \frac{1}{2}(1 \cdot 3 + 3 \cdot 9 + 2 \cdot 12) = 36 - \frac{54}{2} = 36 - 27 = 9$

32. If exactly one of the statements below is false, how many integer values less than 100 are possible for the secret number?

A: Statement B is true and the secret number is less than 31.
B: Statement C is false or the secret number is more than 12.
C: Statement D is true or the secret number is less than 49.
D: Statement A is false and the secret number is more than 28.

There are four sets of truth-values to consider: FTTT, TFTT, TTFT, and TTTF.

For the first case, if A is false (but B is true), the secret number must be ≥ 31. If B is true (and C is true), the secret number must be > 12. If C is true (and D is true), nothing can be said about the secret number. If D is true (and A is false), the secret number must be > 28. Thus, the first case is possible, and allows values from 31 to 99.

For the second case, if A is true (and B is false), something is wrong. This case is impossible. Similarly for the third case if C is false (and D is true), something is wrong and this case is impossible.

For the final case, if A is true (and B is true), the secret number must be < 31. If B is true (and C is true), the secret number must be > 12. If C is true (and D is false), the secret number must be < 49. If D is false (and A is true), nothing can be said about the secret number. Thus, the final case is possible, and allows values from 13 to 30, so that overall any value from 13 to 99 is possible, for an answer of 87.

33. A cube of red plastic is painted blue on the outside, and then it is cut into 125 smaller cubes. How many of these smaller cubes are blue on at least two faces?

The cube is cut into five slices in each direction to make $5^3 = 125$ smaller cubes. 8 of these were the vertices of the original cube, and thus have three blue faces. Cubes that were on the edges of the original will have two blue faces. Each edge has $5 - 2 = 3$ such cubes, and there are 12 edges, for a subtotal of $3 \cdot 12 = 36$ and an answer of $36 + 8 = 44$.

34. Andrea could do the job in 12 hours, and Bernie could do the job in 20 hours. How many minutes, to the nearest minute, would it take for Andrea and Bernie to do the job if they work together?

A's speed is $\frac{1}{12}$ jobs per hour, while B's is $\frac{1}{20}$, so their speed working together will be $\frac{1}{12} + \frac{1}{20} = \frac{5+3}{60} = \frac{8}{60} = \frac{2}{15}$, so it will take them $\frac{15}{2}$ hours to do one job, which is $\frac{15}{2} \cdot 60 = 15 \cdot 30 = 450$ minutes.

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35. Evaluate: $\int_2^5 v\sqrt{v-1}\,dv$

If we let $u = v - 1$, then $v = u + 1$ and $du = dv$, so we can write $\int_1^4 (u+1)u^{\frac{1}{2}}du = \int_1^4 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du = \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\Big|_1^4 = \frac{2}{5}(32-1) + \frac{2}{3}(8-1) = \frac{62}{5} + \frac{14}{3} = \frac{(186+70)}{15} = \frac{256}{15}$.

36. When a group of three boys and three girls goes to the movies, every girl sits next to at least one other girl, and every boy sits next to at least one other boy. How many seating arrangements are possible if Ellie (a girl) sits directly to the right of Frank (a boy)?

If we start with FE, neither of them is sitting next to someone of the same gender, so we can extend to BFEG. Everyone in this arrangement is happy, so let's consider the people who aren't in there yet. They are a boy and a girl, each of which must be next to another person of the same gender, so the final arrangement must be BBFEGG. There are two ways to arrange the boys who aren't Frank, and two for the girls who aren't Ellie, for an answer of $2 \cdot 2 = 4$.

37. If $m(n) = 9n - 1$, evaluate $m^{-1}(-55)$.

If $-55 = 9n - 1$, then $-54 = 9n$ and $-6 = n$.

38. What value of x satisfies the Mean Value Theorem for Derivatives for the function $h(x) = x^2 + 3$ on the interval $[1, 4]$?

The line from (1,4) to (4,19) has a slope of $\frac{19-4}{4-1} = \frac{15}{3} = 5$. $h'(x) = 2x = 5$ is true at $x = \frac{5}{2}$.

39. Your gluttonous meal cost a total of \$123.75 after the 5% sales tax and your generous 20% tip. If the tip was calculated based on the pre-tax amount, how much (in dollars to the nearest hundredth) did you tip your server?

We had a bill, they added 5%, then we added 20% of the same amount, so 123.75 is 125% of the original bill, so our tip was $\frac{20}{125} \cdot 123.75 = \frac{4}{25} \cdot 123.75 = 4 \cdot 4.95 = 19.80$.

40. What is the missing term of the sequence 2187, 1782, 1458, 1336, 972, 890, 648, 444, ___, …?

The differences are ~400, ~300, ~100, ~400, ~100, ~200, ~200, the differences of these are ~100, ~200, ~ -300, ~300, ~ -100, ~0. It doesn't seem like this is leading anywhere, so let's try alternating terms. Those differences are ~700, ~500, ~300 (maybe quadratic?), and ~400, ~400, and ~400 (maybe arithmetic!). Assume it's alternating; we need to examine 2187, 1458, 972, 648, ___. These are all multiples of 9, so we can examine 243, 162, 108, 72, __. These are still all multiples of 9, so we can examine 27, 18, 12, 8, __. Aha, this is geometric with $r = \frac{2}{3}$, for a final term of $8 \cdot \frac{2}{3} = \frac{16}{3}$, which would really be $\frac{16}{3} \cdot 81 = 16 \cdot 27 = 432$.