2018 Four-by-Four Competition
Solutions

1. **Simplify by rationalizing the denominator:** $\frac{24}{8+\sqrt{61}}$

   $\frac{24}{8+\sqrt{61}} = \frac{24}{8+\sqrt{61}} \cdot \frac{8-\sqrt{61}}{8-\sqrt{61}} = \frac{24(8-\sqrt{61})}{64-61} = \frac{24(8-\sqrt{61})}{3} = 8(8-\sqrt{61}) = 64 - 8\sqrt{61}$

2. **What is the value of the missing term in the sequence 60, 80, 132, 120, 204, 180, 276, 270, ___, 405, 420, …**

   When a sequence doesn't make sense, it's frequently two (or more) sequences that have been interspersed. The 80, 120, 180, 270, 405 sequence is geometric with ratio $\frac{120}{80} = \frac{12}{8} = \frac{3}{2}$, leaving 60, 132, 204, 276, ___, 420, … This is arithmetic with difference 72, making the missing term $276 + 72 = 420 - 72 = 348$.

3. **If $s \blacksquare t = -7s^2t - \frac{5s}{t} + 6t^2$, evaluate $6 \blacksquare 2$.**

   $6 \blacksquare 2 = -7 \cdot 6^2 \cdot 2 - \frac{5\cdot 6}{2} + 6 \cdot 2^2 = -504 - 15 + 24 = -495$

4. **A right triangle has an angle measuring $60°$ and a perimeter of $3\sqrt{6} + 3\sqrt{2}$ m. What is the area, in square meters, of the triangle?**

   It's a 30-60-90 triangle, so the sides are in the ratio $1:\sqrt{3}:2$. That means that two of them add up to a multiple of 3, and the other side is that sum divided by $\sqrt{3}$. Thus, the multiple of 3 is the $3\sqrt{6}$, so that the other side will be $3\sqrt{2}$, and the area is $\frac{1}{2}\sqrt{6} \cdot 3\sqrt{2} = \frac{3}{2}\sqrt{12} = 3\sqrt{3}$.

5. **What are the coordinates, in the form (x, y), of the vertex of the parabola with equation $y = 5x^2 + 4x - 1$?**

   The vertex is on the axis of symmetry, which is $x = -\frac{b}{2a} = -\frac{4}{2\cdot 5} = -\frac{2}{5}$, so $y = 5\left(-\frac{2}{5}\right)^2 + 4\left(-\frac{2}{5}\right) - 1 = \frac{4}{5} - \frac{8}{5} - \frac{5}{5} = -\frac{9}{5}$, for an answer of $\left(-\frac{2}{5}, -\frac{9}{5}\right)$.

6. **What is the measure, in degrees, of the smaller angle between the hour and minute hands of a standard 12-hour analog clock at 3:50 AM?**

   If the hour hand were on the 3 (it's actually further), it would be $3 \cdot 30 = 90°$ from 12. The minute hand is on the 10, which is $2 \cdot 30 = 60°$ from the 12. The hour hand is $\frac{5}{6} \cdot 30 = 25$ degrees past the three, for an answer of $90 + 60 + 25 = 175°$.

7. **How many subsets of the nine smallest counting numbers contain exactly two prime numbers and exactly three even numbers?**

   Of the numbers 1-9, the primes are 2, 3, 5, and 7, and the evens are 2, 4, 6, and 8, so 2 is both and 1 & 9 are neither. Thus, we could have 2 along with one other prime (3 ways) and two other evens (3 ways, for a subtotal of 9 ways), or we could NOT have 2 along with two other primes (3 ways) and the three other evens (1 way, for a subtotal of 3 ways). This gives us $9 + 3 = 12$ ways to use the evens and primes, but each of 1 and 9 can choose to be in or out, for an answer of $12 \cdot 2^2 = 12 \cdot 4 = 48$.

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8. What is the largest number less than 1000 that leaves a remainder of 6 when divided by 70 and 1 when divided by 45?

The number will be of the form $6 + 70k = 1 + 45m$, which becomes $5 = 45m - 70k$, then $1 = 9m - 14k$. The multiples of 14 are 14, 28, 42, 56, 70, 84, 98, …, with 98 being one less than a multiple of 9, so $k = 7$ and $m = 11$, meaning that $6 + 70 \cdot 7 = 1 + 45 \cdot 11 = 496$ satisfies the conditions, but is it close enough to 1000? We can add the LCM of 45 and 70 as much as we'd like to get additional solutions… The LCM is $5 \cdot 9 \cdot 14 = 9 \cdot 70 = 630$, so 496 is our answer.

9. What is the slope of the line through the point $(-1, 1)$ and perpendicular to the line $6x - 3y = 2$?

The point is irrelevant. The slope of the given line is $m = -\frac{a}{b} = -\frac{6}{-3} = 2$, so the slope of any perpendicular line is $-\frac{1}{m} = -\frac{1}{2}$.

10. What is the sum of the first 14 terms of the arithmetic sequence with first term 4 and common difference 7?

The 14th term is $4 + 13 \cdot 7 = 4 + 91 = 95$, so that each of the $\frac{14}{2} = 7$ "outer pairs" adds up to $95 + 4 = 99$, for an answer of $7 \cdot 99 = 693$.

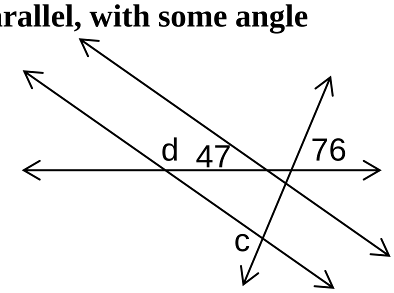
11. A math club's membership consists of 6 boys and 7 girls. Their bylaws state that they must have a President, a Vice-President, and two Representatives to the ASB, and that these four people must include at least one boy and at least one girl. In how many ways could these offices be filled according to these rules?

Ignoring gender, there are 13 possible Presidents, 12 possible VPs for each President, and $\binom{11}{2} = 55$ possible sets of Representatives, for a total of $55 \cdot 156 = 8580$ options. However, we can't use all boys or all girls. There are $6 \cdot 5 \cdot 4 \cdot 3 \div 2 = 180$ all-boy options and $7 \cdot 6 \cdot 5 \cdot 4 \div 2 = 420$ all-girl options, for an answer of $8580 - 180 - 420 = 7980$.

12. A dodecagon has U interior angles measuring $120°$, and all the other interior angles measure $V°$. What value of V corresponds to the largest possible value of U?

A dodecagon has interior angles that sum to $180(12 - 2) = 1800°$, which gives an average angle of $\frac{1800}{12} = 150°$. To have a lot of $120°$ angles, we need the other angles to be as large as possible, which is just shy of $360°$. One $360°$ angle would leave $1440°$ for eleven angles, which is more than $120°$ each. Two $360°$ angles would leave $1080°$ for ten angles, which is less than $120°$! This means we can have ten $120°$ angles totaling $1200°$, which leaves $600°$ for the other two angles, making them $300°$ each.

13. The figure to the right shows four lines, two of which are parallel, with some angle measures given in degrees. What is the value of $c + d$?



c and d share a diagonal line. If you picture them at the same vertex, the remaining angle would be the supplement of $76°$, so that $c + d = 360 - (180 - 76)) = 180 + 76 = 256$.

14. Evaluate: $\frac{9!\cdot 11!\cdot 7!}{8!\cdot 14!\cdot 5!}$

$\frac{9!\cdot 11!\cdot 7!}{8!\cdot 14!\cdot 5!} = \frac{9\cdot 7\cdot 6}{14\cdot 13\cdot 12} = \frac{9}{2\cdot 13\cdot 2} = \frac{9}{52}$

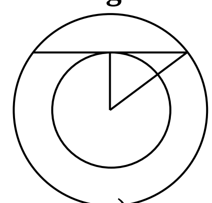
15. What is the sum of the factors of 975?

$975 = 25 \cdot 39 = 3 \cdot 5^2 \cdot 13$. The sum of the factors will thus be $(1+3)(1+5+25)(1+13) = 4 \cdot 31 \cdot 14 = 31 \cdot 56 = 1736$.

16. My pocket contains 21 coins, each of which is either a quarter, dime, nickel, or penny. If the total value of these coins is $3.04, what is the largest possible number of dimes in my pocket?

The number of pennies is at least 4, leaving 17 other coins worth 300 cents. The average value of these 17 coins is ~ 17.5 cents implying that a roughly even split of quarters and dimes could work. If we try 8 quarters, they're worth 200 cents, requiring 10 dimes, which is too many coins. 9 quarters are worth 225 cents, requiring 7 dimes and a nickel, which works, giving an answer of 7.

17. Two concentric circles are drawn so that the area of the annular region between them is 75π m². What is the length, in meters, of a chord of the larger circle that is tangent to the smaller circle?



The first to the right shows that the square of half the chord is equal to $R^2 - r^2$, which is cool because the area in question is $75\pi = \pi(R^2 - r^2)$, so the length of the chord is $2\sqrt{75} = 2 \cdot 5\sqrt{3} = 10\sqrt{3}$.

18. If $w(x) = 4x - 1$, $y(z) = -2z + 6$, and $b(c) = -5c - 4$, evaluate $y\left(w\left(b(-5)\right)\right)$.

$y\left(w\left(b(-5)\right)\right) = y\left(w(21)\right) = y(83) = -160$

19. An unfair coin has a $\frac{25}{49}$ probability of coming up heads. When it is flipped seven times, what is the ratio, expressed as a fraction, between the probability of getting exactly three heads and the probability of getting exactly three tails?

The probability of three heads is $\binom{7}{3}\left(\frac{25}{49}\right)^3\left(\frac{24}{49}\right)^4$, and the probability of three tails is $\binom{7}{4}\left(\frac{25}{49}\right)^4\left(\frac{24}{49}\right)^3$. The ratio involves a lot of cancelling $\left(\binom{7}{3} = \binom{7}{4}\right)$, yielding $\frac{24}{25}$.

20. What is the volume of the solid generated when the area between $y = x^2$ and $y = 13 - (x-1)^2$ is rotated about the line $x = 5$?

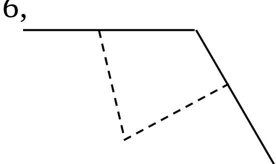
Setting the y's equal to one another, we get $x^2 = 13 - x^2 + 2x - 1$, then $2x^2 - 2x - 12 = 0$, then $x^2 - x - 6 = 0$, which factors to $(x-3)(x+2) = 0$ with roots of 3 and -2. Instead of using washers or shells, the fact that the two parabolas are equally steep means we can find the centroid to be at $x = \frac{0+1}{2} = \frac{3+(-2)}{2} = \frac{1}{2}$ and use the centroid-area method for the volume of rotation. The area is $\int_{-2}^{3}(12 + 2x - 2x^2)dx = \left[12x + x^2 - \frac{2}{3}x^3\right]_{-2}^{3} = 12\big(3-(-2)\big) + (9-4) - \frac{2}{3}\big(27-(-8)\big) = 60 + 5 - \frac{70}{3} = \frac{125}{3}$. We can find the volume of rotation by multiplying by the distance the centroid moves, which is $2\pi r$, where $r = 5 - \frac{1}{2} = \frac{9}{2}$, for an answer of $\frac{125}{3} \cdot 9\pi = 125 \cdot 3\pi = 375\pi$.

21. Moldessa and Napstablook see one another at the same moment when they are 2400 meters apart and immediately rush towards one another. If Napstablook floats at a speed of 2 meters per second (mps) and Moldessa oozes at a speed of 1 mps, how many minutes will it take them to reach one another?

Their combined speed is $2 + 1 = 3$, so it will take them $\frac{2400}{3} = 800$ seconds, which is $\frac{800}{60} = \frac{80}{6} = \frac{40}{3}$ minutes.

22. Express the base-10 numeral 539_{10} as a base-4 numeral.

From right to left, the digits in base 4 represent $4^0 = 1$, $4^1 = 4$, $4^2 = 16$, 64, 256, etc. 539 will have two 256s, leaving $539 - 512 = 27$. There are no 64s in 27, but there is one 16, leaving $27 - 16 = 11$. There are two 4s in 11, leaving $11 - 8 = 3$, for an answer of 20123.



23. A folding screen has two sections that are each four feet wide. If this screen is placed in the corner of a (regular) hexagonal room (see figure to the upper right), what is the largest area, in square feet, that it can separate from the rest of the room?

The symmetry of the situation implies that the ends of the screen should be the same distance from the corner. If we consider three hexagonal rooms surrounding the corner, each with a screen, the three screens would form a hexagon. This hexagon's area will be maximized when it is a regular hexagon, and the area behind each screen is at its maximum, which will be one-third of the hexagon, or two equilateral triangles, for an answer of $2 \cdot \frac{4^2\sqrt{3}}{4} = 8\sqrt{3}$.

24. What is the area of the ellipse with equation $5x^2 + y^2 - 8x + 7y = 8$?

The area of an ellipse is πab, where a and b are the lengths of the semi-axes, which we can get by putting the equation in standard form. $5x^2 + y^2 - 8x + 7y = 8$ becomes $5\left(x^2 - \frac{8}{5}x\right) + y^2 + 7y = 8$, then $5\left(x - \frac{4}{5}\right)^2 + \left(y + \frac{7}{2}\right)^2 = 8 + \frac{16}{5} + \frac{49}{4} = \frac{160+64+245}{20} = \frac{469}{20}$, and finally $\frac{\left(x-\frac{4}{5}\right)^2}{\frac{469}{100}} + \frac{\left(y+\frac{7}{2}\right)^2}{\frac{469}{20}} = 1$. Thus $a = \sqrt{\frac{469}{100}}$ and $b = \sqrt{\frac{469}{20}}$, so that the area is $\pi\sqrt{\frac{469}{100}}\sqrt{\frac{469}{20}} = \frac{469\pi}{20\sqrt{5}} = \frac{469\pi\sqrt{5}}{100}$.

25. When the centers of the faces of a regular dodecahedron are used as the vertices of a new regular polyhedron, what is the name for that polyhedron?

There are 12 faces on a dodecagon, so there are 12 vertices on the new polyhedron, which makes it an icosahedron. Polyhedrons having this relationships are called “duals”.

26. How many prime numbers are there between 70 and 100?

71 is prime, as is 73, 75 is not (5), nor 77 (7), 79 is, 81 is not (9), 83 is, 85 is not (5), 87 is not (3), 89 is, 91 is not (7), nor is 93 (3), nor is 95 (5), 97 is, and 99 is not (9). The primes are 71, 73, 79, 83, 89, and 97, for an answer of 6.

27. Last night at charity Casino Night, I saw the new game Rollow. Players pay \$5 to roll two dice. They win \$2 for each 1 they roll, \$1 for each 2 they roll, and an additional bonus of \$40 if they roll two 1s. What is a player’s expected loss, in dollars rounded to the nearest hundredth (cent), when they play this game?

Consider a red die and a blue die. There is $\frac{1}{6}$ chance the red die shows a 1, which contributes $\frac{1}{6} \cdot 2 = \frac{1}{3}$ to the expected value. Similarly, the 2 contributes $\frac{1}{6}$, and similar analysis of the blue die gives $\frac{1}{3}$ and $\frac{1}{6}$. Finally, there is a $\frac{1}{36}$ chance we get an _additional_ \$40, for an answer of $\frac{1}{3} + \frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{40}{36} - 5 = 1 + \frac{40}{36} - 5 = \frac{10}{9} - 4 = -\frac{26}{9} \approx 2.89$.

28. Evaluate: $\lim\limits_{n\to 0} \frac{7\sin^2 5n \cos 9n}{6n^2}$

Because this evaluates to $\frac{0}{0}$, L’Hopital’s Rule says we need to take derivatives. The bottom is easy: $12n$. The top is more complicated, so we’re only going to focus on the “problem” term, the sin which is making things 0. We’ll only address the term where we’re making progress getting rid of the sin, as this will be the first term to be non-zero. The product rule on the top will produce two terms, and we’ll focus on the one that takes the derivative of the sin, getting $7 \cdot 2 \sin 5n \cos 5n \cdot 5 \cdot \cos(9n) = 70 \sin 5n \cos 5n \cos 9n$. Sadly, we still get $\frac{0}{0}$ at this point, so we need to go further. The bottom becomes 12, and one term of the top will be $70 \cos 5n \cdot 5 \cdot \cos 5n \cos 9n = 350 \cos^2 5n \cos 9n$. There are other terms on top, but they’ll still be 0. Now we get $\frac{350}{12} = \frac{175}{6}$.

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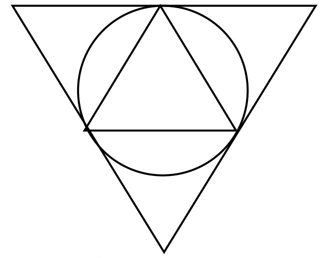
29. What are the coordinates, in the form (x, y), of the rightmost x-intercept of the parabola with equation $y = x^2 + 2x$?

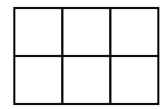
$y = x(x + 2)$ has roots of 0 and -2, so the rightmost x-intercept is $(0,0)$.

30. A circle has both an inscribed equilateral triangle and a circumscribed equilateral triangle. What is the ratio, as a fraction, of the area of the smaller triangle to the area of the larger triangle?



The diagram to the right shows that the answer is $\frac{1}{4}$.

31. In the Radical Rectangle to the right, six distinct positive integers must be placed in the cells (one number per cell, in any order) so that the sums of the three numbers in each row are equal, and the sums of the two numbers in each column are equal. What is the smallest possible sum of all six numbers?



If the row sum is b, then the sum of all six numbers is $2b$. If the column sum is c, then the sum of all six numbers is $3c$. Thus, the sum of all six numbers must be a multiple of both 2 and 3, so a multiple of 6. The sum of the numbers 1-6 is 21, so 24 (the next multiple of 6) is the smallest possible sum of the six numbers. And yes, it works, for example 1, 5, 6, 7, 3, 2.

32. Express $\frac{9.4\times10^6+6.224\times10^9}{9\times10^4}$ in scientific notation rounded to four significant figures.

$\frac{9.4\times10^6+6.224\times10^9}{9\times10^4} = \frac{6233.4\times10^6}{9\times10^4} = \frac{6233.4\times10^2}{9} = \frac{623340}{9} = 69260 = 6.926\times10^4$

33. What is the period, in radians, of $2\sin^3 8k + 9\cos^2 6k$?

The period of both sin & cos is 2π. The period of $\sin^3$ is also 2π, but the period of $\cos^2$ is only π, because the negative parts of the cos graph become positive. Thus, the period of $\sin^3 8k$ is $\frac{2\pi}{8} = \frac{\pi}{4}$, and the period of $\cos^2 6k = \frac{\pi}{6}$. The period of their sum will be the least common multiple of their periods, which is $\frac{\pi}{2}$.

34. What is the remainder when 3975 is divided by 74?

75 goes into 3000 40 times, 900 12 times, and 75 one time, for a total of $40 + 12 + 1 = 53$ times. 53 74's should be 53 less than 3975, making our answer 53.

35. What is the sum of the first ten terms of the Fibonacci Sequence, the first two terms of which are 1 and 1?

The terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, … The sum of the first term is 1, of the first two terms is $1 + 1 = 2$, then $2 + 2 = 4$, $4 + 3 = 7$, $7 + 5 = 12$, etc. Notice that the sum of the first n terms is one less than the $n + 2$th term! Thus, the sum of the first ten terms should be one less than the twelfth term, for an answer of $144 - 1 = 143$.

36. If $\log_5 2 = d$, express $\log\frac{5}{4}$ in terms of d and without logarithms.

$\log\frac{5}{4} = \log_{10}\frac{5}{4} = \log_{10}\frac{10}{8} = \log_{10} 10 - \log_{10} 8 = 1 - 3\log_{10} 2 = 1 - 3\left(\frac{\log_5 2}{\log_5 10}\right) = 1 - 3\left(\frac{d}{\log_5 5+\log_5 2}\right) = 1 - 3\left(\frac{d}{1+d}\right) = \frac{1+d}{1+d} - \frac{3d}{1+d} = \frac{1-2d}{1+d}$

37. What number is 253 more than twice 984?

$253 + 2 \cdot 984 = 253 + 1968 = 2221$

38. What is the average value of $9m^2 - 4$ on the interval $[-2, 1]$?

The average value is $\frac{\int_{-2}^{1}(9m^2-4)dm}{1-(-2)} = \frac{1}{3}\int_{-2}^{1}(9m^2 - 4)dm = \frac{1}{3}[3m^3 - 4m]_{-2}^{1} = \frac{1}{3}\left[3(1^3 - (-2)^3) - 4\big(1 - (-2)\big)\right] = \frac{1}{3}[3(1+8) - 4(3)] = \frac{1}{3}(27 - 12) = \frac{15}{3} = 5$.

39. If seven Octopuses are equivalent to one Peacock, nine Quetzals are equivalent to six Rats, and one Octopus is equivalent to eight Rats, how many Peacocks are equivalent to 504 Quetzals?

504 Quetzals are equivalent to $504 \cdot \frac{6}{9}$ Rats, which are equivalent to $504 \cdot \frac{6}{9} \cdot \frac{1}{8}$ Octopuses, which are equivalent to $504 \cdot \frac{6}{9} \cdot \frac{1}{8} \cdot \frac{1}{7}$ Peacocks, for an answer of $\frac{504 \cdot 6}{9 \cdot 8 \cdot 7} = \frac{504 \cdot 2}{3 \cdot 8 \cdot 7} = \frac{63 \cdot 2}{3 \cdot 7} = \frac{9 \cdot 2}{3} = 3 \cdot 2 = 6$.

40. What is the equation, in the form $x = f(y) = g(z)$, of the line through the points $(-7, 3, -8)$ and $(3, -2, -2)$?

The vector between the two points is $\langle 3 - (-7), -2 - 3, -2 - (-8)\rangle = \langle 10, -5, 6\rangle$, which allows us to write $x = -7 + 10t$, $y = 3 - 5t$, and $z = -8 + 6t$. Solving for t, we get $t = \frac{x+7}{10} = \frac{y-3}{-5} = \frac{z+8}{6}$. We need to solve for x, so we next get $x + 7 = -2y + 6 = \frac{5z+40}{3}$, and finally $x = -2y - 1 = \frac{5z+19}{3}$.