2017 Four-by-Four Competition
Solutions

1. **How many real roots does $3v^2 - 4v - 5 = 0$ have?**

   The discriminant is $b^2 - 4ac = (-4)^2 - 4 \cdot 3(-5) = 16 + 60 = 76 > 0$ so there are two real roots.

2. **What is the largest possible area, in square meters, of a triangle with vertices on a circle with a radius of 4 m?**

   Consider one side to be fixed – the maximum area for that triangle will be when the height relative to that side is maximized. Now consider a different side of that semi-maximized triangle – again the height should be maximized, and when we repeat this process we'll eventually create an equilateral triangle. If the radius is 4, the side of the triangle will be $4\sqrt{3}$ due to 30-60-90 triangles. The area of an equilateral triangle is $A = \frac{s^2\sqrt{3}}{4} = \frac{(4\sqrt{3})^2\sqrt{3}}{4} = \frac{48\sqrt{3}}{4} = 12\sqrt{3}$.

3. **In a right triangle with legs measuring 8 m and 4 m, what is the cosecant of the smallest angle?**

   The smallest angle is opposite the 4, and the cosecant is the reciprocal of the sine, so will be the hypotenuse over the opposite side. The hypotenuse will be $\sqrt{8^2 + 4^2} = 4\sqrt{2^2 + 1^2} = 4\sqrt{5}$, so the answer will be $\frac{4\sqrt{5}}{4} = \sqrt{5}$.

4. **What is the sum of the first fifteen terms of an arithmetic sequence with first term forty and common difference seven?**

   The first term is 40 and the fifteenth term is $40 + 14 \cdot 7 = 40 + 98 = 138$. Outer pairs will sum to $40 + 138 = 178$, for a sum of $\frac{15 \cdot 178}{2} = 15 \cdot 89 = 1335$.

5. **What are the coordinates, in the form (x, y), of the y-intercept of the line $9x - 2y = 54$?**

   The y-intercept is where $x = 0$, so we can write $9 \cdot 0 - 2y = 54$, then $-2y = 54$, giving $y = -27$ for an answer of $(0, -27)$.

6. **Two concentric circles have radii of 7 m and 4 m. What is the length, in meters, of a chord of the larger circle that is tangent to the smaller circle?**

   Half the chord will be the other leg of a right triangle with a hypotenuse of 7 and an other leg of 4, so the chord's length will be $2\sqrt{7^2 - 4^2} = 2\sqrt{49 - 16} = 2\sqrt{33}$.

7. **How many positive integers less than 70 are prime?**

   Knowing there are 25 primes less than 100, we can work backward to 70. 97 is prime, as are 89, 83, 79, 73, and 71, for an answer of $25 - 6 = 19$.

8. **Express in simplest radical form: $\sqrt{7632}$**

   $\sqrt{7632} = 2\sqrt{1908} = 4\sqrt{477} = 12\sqrt{53}$

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9. Evaluate the matrix product: $\begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \end{bmatrix} \begin{bmatrix} \mathbf{4} & \mathbf{5} \\ \mathbf{-6} & \mathbf{7} \\ \mathbf{8} & \mathbf{-9} \end{bmatrix}$

Matrix multiplication goes across the first matrix and down the second, for a first element of $1 \cdot 4 + (-2)(-6) + 3 \cdot 8 = 4 + 12 + 24 = 40$ and a second element of $1 \cdot 5 + (-2)7 + 3(-9) = 5 - 14 - 27 = -36$, for an answer of $\begin{bmatrix} 40 & -36 \end{bmatrix}$.

10. In the new casino game of Headers, you pay $3 to flip five coins. You receive one dollar back for each coin that shows heads. What is the expected value, in dollars rounded to the nearest cent, of your loss? Note, an expected loss would be expressed as a positive number, while an expected gain would be negative.

You could analyze each outcome (5 Heads, 4 Heads, etc.), but you can also consider that each coin is expected to show half a head, so the expected number in five flips is $\frac{5}{2} = 2.50$, for an answer of $3 - 2.50 = 0.50$.

11. A cube of solid white plastic with edges measuring 7 m is painted blue on five faces, then the cube is sliced into cubes that are 1 m on each edge. How many of these smaller cubes have exactly one blue face?

To have one blue face, a sub-cube might have been in the interior of a blue face or on the edge of a white face (but not a corner). For the first case, there are five faces to consider, and the interior of each of those faces measures $7 - 2 = 5$ by $7 - 2 = 5$, for a subtotal of $5 \cdot 5 \cdot 5 = 5^3 = 125$ sub-cubes. For the second case, there are four edges to consider, each of which has $7 - 2 = 5$ sub-cubes, for a subtotal of $4 \cdot 5 = 20$ and a final answer of $125 + 20 = 145$.

12. Bart's age is the sum of Lisa and Maggie's ages. Five years ago, Bart's age was twice Lisa's age, which was twice Maggie's age. How old is Bart now?

The easiest way to approach this is probably to make a list of ages they might have had five years ago: 4, 2, 1; 8, 4, 2; 12, 6, 3; 16, 8, 4; 20, 10, 5; 24, 12, 6; etc. Now consider what Bart's age might be now, vs. the sum of the others' ages: 9 vs. 13, 13 vs. 16, 17 vs. 19, 21 vs. 22, 25 vs. 25 (yay!). Bart is 25 now.

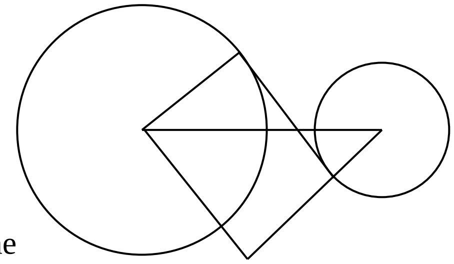
13. How many months are in $3\frac{3}{4}$ decades?

There are ten years in a decade and twelve months in a year, so our answer will be $3\frac{3}{4} \times 10 \times 12 = \frac{15}{4} \times 120 = 15 \times 30 = 450$.

14. Express the domain of $p(q) = \frac{\sqrt{4-q^2}}{q-2}$ in interval notation. Assume the domain and range are both subsets of the real numbers.

The domain in this case is the values that q can have, which generally is any number that won't "break" the function. One way to break a function is to divide by zero, which is a possibility here, so $q - 2 \neq 0$, giving $q \neq 2$. Another way to break a real-valued function is to take the square root of a negative number, so $4 - q^2 \geq 0$, giving $-2 \leq q \leq 2$. For both constraints to be true, we'd write $-2 \leq q < 2$, which is $[-2,2)$ in interval notation.

15. Two circles with radii of 2 m and 4 m have their centers 7 m apart. A line segment is drawn from one circle to the other, tangent to both. What is the shortest, in meters, such a line segment can be?



There are two possible tangent lengths; the "external tangent" will be shorter than 7 by some amount, while the "internal tangent" will be longer than $7-4-2=1$ by some amount, and thus is likely the smaller of the two (in fact, this is always the case). In the figure to the right, a congruent segment parallel to the internal tangent forms the leg of a right triangle with a hypotenuse of 7 and an other leg of $4+2=6$, giving a length of $\sqrt{7^2-6^2}=\sqrt{13}$.

16. A data set has a median of 5689. When two elements of 8359 are added to the data set, the new median is 8345. When six additional elements of 2380 are added to the data set, the new median is 4579. When four more elements of 67925 are added to the data set, what is the new median?

Adding two elements above the median will raise the median by one element. Adding six elements below this median will lower the median by three elements. Adding four elements above this median will raise it by two elements, which is back to the original value, 5689.

17. When Mr. E asks his students to find the roots of an equation of the form $g^2+Bg+C=0$, Hao miscopies the value of B to get roots of -24 and $-\frac{1}{2}$, while Iker miscopies the value of C to get roots of -3 and 11. What are the roots of Mr. E's original equation?

Hao's value of C must have been $(-24)\left(-\frac{1}{2}\right)=12$, while Iker's value of B must have been $-(-3+11)=-8$, making Mr. E's equation $g^2-8g+12=0$, which factors to $(g-2)(g-6)=0$ with roots of 2 and 6.

18. What is the area, in square meters, of an isosceles triangle with sides measuring 5 m and 13 m?

The triangle must be 5-13-13, and the median to the 5 side divides the triangle into two congruent right triangles. $h=\sqrt{13^2-\left(\frac{5}{2}\right)^2}=\sqrt{169-\frac{25}{4}}=\frac{\sqrt{676-25}}{2}=\frac{\sqrt{651}}{2}$, so $A=2\cdot\frac{1}{2}\cdot 5\cdot\frac{\sqrt{651}}{2}=\frac{5\sqrt{651}}{2}$.

19. A bag contains four red, seven green, and three blue marbles. If three marbles are chosen randomly, what is the probability that exactly two of them are the same color?

They could be RRO(ther), GGO, or BBO, and there are $4c2\cdot 10c1=6\cdot 10=60$, $7c2\cdot 7=21\cdot 7=147$, and $3c2\cdot 11=3\cdot 11=33$ ways for those to happen, respectively. There are $14c3=\frac{14\cdot 13\cdot 12}{3\cdot 2}=7\cdot 13\cdot 4=91\cdot 4=364$ total ways to draw three marbles, for an answer of $\frac{60+147+33}{364}=\frac{240}{364}=\frac{60}{91}$.

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20. What is the missing term in the sequence beginning 4, 5, 12, 13, 36, 35, 108, 71, 324, ___, …?

The odd terms are 4, 12, 36, 108, 324, which is a geometric sequence with common ratio 3, but this doesn't explicitly help us find the missing term. The even terms are 5, 13, 35, 71, ___. The differences in this sequence are 8, 22, and 36, which form an arithmetic sequence with common difference 14. The next term of this sequence of differences would be $36 + 14 = 50$, so that the next term of the sequence of even terms would be $71 + 50 = 121$.

21. If $\boldsymbol{u(v) = (v+2)^3 \sin(4v)}$, evaluate $\boldsymbol{u'(0)}$.

$u'(v) = 3(v+2)^2 \sin(4v) + (v+2)^3 \cos(4v)\,(4)$, so $u'(0) = 3 \cdot 2^2 \sin(0) + 2^3 \cos(0)\,(4) = 3 \cdot 4 \cdot 0 + 8 \cdot 1 \cdot 4 = 0 + 32 = 32$.

22. Express $\boldsymbol{6.4\overline{8}}$ as an improper fraction.

$x = 6.4\overline{8}$, $10x = 64.\overline{8}$, and $100x = 648.\overline{8}$. Subtracting the latter two gives $90x = 584$, for an answer of $\frac{584}{90} = \frac{292}{45}$.

23. When two cards are drawn from a standard 52-card deck without replacement, what is the probability that the first card is a red face card (J, Q, or K), and that the second is a Spade?

The probability will be $\frac{6}{52} \cdot \frac{13}{51} = \frac{2}{4} \cdot \frac{1}{17} = \frac{1}{34}$.

24. What is the equation of the axis of symmetry of the graph of $\boldsymbol{y = 2x^4 - 24x^3 + 109x^2 - 222x + 168}$?

If this has an axis of symmetry, it's of the form $x = c$, and we might want to consider terms like $(x-c)^2$ and $(x-c)^4$. Looking at the first two terms, they could be generated by $2(x-3)^4 = 2(x^4 - 12x^3 + 54x^2 - 108x + 81) = 2x^4 - 24x^3 + 108x^2 - 216x + 162$. This is looking so good that I'd be happy to answer $x = 3$ right now, but just for completeness, we're still missing $x^2 - 6x + 6 = (x-3)^2 - 3$, and now we're certain that $x = 3$ is the axis of symmetry.

25. What are the coordinates, in the form $\boldsymbol{(x, y)}$, of the vertex of the parabola $\boldsymbol{y = 8x^2 - 9x - 4}$?

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{-9}{2 \cdot 8} = \frac{9}{16}$, and substituting this gives $y = 8\left(\frac{9}{16}\right)^2 - 9\left(\frac{9}{16}\right) - 4 = -\frac{81}{32} - \frac{128}{32} = -\frac{209}{32}$, for an answer of $\left(\frac{9}{16}, -\frac{209}{32}\right)$.

26. In a triangle with sides measuring 4 m, 5 m, and 6 m, what is the length, in meters, of the median to the longest side?

Using Stewart's Theorem, we can write $4^2 \cdot 3 + 5^2 \cdot 3 = t^2 \cdot 6 + 3^2 \cdot 6$, which becomes $16 + 25 = 2t^2 + 18$, giving $23 = 2t^2$, then $\frac{23}{2} = t^2$, and finally $t = \sqrt{\frac{23}{2}} = \frac{\sqrt{46}}{2}$.

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27. In the cryptarithm below, each instance of a letter represents the same digit 0-9, and different letters represent different digits. E.g. if one A represents a 1, all A's represent 1's and B's cannot represent 1's. What is the largest possible value of the five-digit number ABCDE? $AB \times BC = DEA$?

We can immediately determine that $B \times AB$ is a two-digit number, and by considering $BC \times AB = DEA$, $A \times BC$ is also a two-digit number. For a large answer, we'd like A to be large, then B, then C, etc. A cannot be 9, for in the latter case 9×12 is the smallest possible sub-problem. If A were 8, BC could be 12 to produce a two-digit product, and 1×81 is also two-digits, but if B is 1 then $B \times C$ cannot be A, it would be C, so neither B nor C can be 1 (or 0, which we'd neglected to consider earlier). Perhaps B is 2, in which case A might be 4. In $42 \times 2C = DE4$, $2 \times C$ needs to end in 4, but C cannot duplicate 2, but might be 7, however this results in a four-digit product rather than a three-digit one. Now we must consider $A = 3$, in which case B can still only be 2, giving $32 \times 2C = DE3$, which is obviously impossible. If A is 2, B could be as large as 4, giving $24 \times 4C = DE2$, so that $C \times 4$ ends in 2, making C 3 or 8. 8 is larger, so we'll try it first. It turns out to be too large, so we'll look at $C = 3$, which is still too large. Now we must consider $A = 2$ and $B = 3$, with $3 \times C$ ending in 2, so that C is 4, which works, giving $23 \times 34 = 782$, for an answer of 23478.

28. What is the sum of the positive two-digit even numbers that do not contain a 2?

There are $99 - 9 = 90$ two-digit numbers, half of which (45) will be even. These evens range from 10 to 98, so adding by outer pairs gives a sum of $\frac{45(10+98)}{2} = \frac{45 \cdot 108}{2} = 45 \cdot 54 = 2430$. The numbers containing a 2 are 20 to 28 (5) as well as 12 to 92 (9), with 22 being counted twice. 20 to 28 add up to $\frac{5(20+28)}{2} = 5 \cdot 24 = 120$, and 12 to 92 add up to $\frac{9(104)}{2} = 9 \cdot 52 = 468$. Thus, our answer is $2430 - 120 - 468 + 22 = 2452 - 588 = 1864$.

29. A triangle has two sides measuring 39 m and 56 m, subtending an angle of $135°$ between them. What is the area, in square meters, of this triangle?

$A = \frac{1}{2}ab \sin C = \frac{1}{2} \cdot 39 \cdot 56 \cdot \frac{\sqrt{2}}{2} = 39 \cdot 14\sqrt{2} = 546\sqrt{2}$

30. Simplify in terms of $\boldsymbol{i}\left(= \sqrt{-1}\right)$: $\boldsymbol{i^7(4 - i)^2 + i^{-3}(7 + i)}$

This becomes $(-i)(15 - 8i) + i(7 + i) = -15i - 8 + 7i - 1 = -8i - 9 = -9 - 8i$.

31. A prospector needs to visit both the river $(y - x = 10)$ and his claim $(12, -5)$ (in either order) on the way to the town of Origin. If he is currently at $(9, 15)$, what is the shortest distance he can travel?

If you make a rough sketch, it's clear that going to the river first is the shorter route. The last leg from the claim to Origin is $\sqrt{12^2 + 5^2} = \sqrt{169} = 13$. To minimize the total distance traveled from his current position to the right to the claim, we can reflect his current position across the river to $(5,19)$ and then travel straight from here to $(12, -5)$, which is a distance of $\sqrt{7^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$, for an answer of $13 + 25 = 38$.

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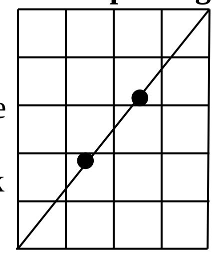
32. A 3x4x5 box is completely filled with 1x1x2 blocks. After this, a hole is drilled between two opposite vertices. What is the smallest number of blocks that this hole could pass through? Note: passing through only an edge or only a vertex does not count as passing through a block.



Drawing a 4x5 grid representing the view from above, the diagonal passes between unit squares as shown, as well as vertically at the dots. To minimize the number of blocks it passes through, we'll put a North-South block in the lower left corner, an Up-Down block through the first dot, a Right-Left block just north of this, an Up-Down block through the second dot, and a North-South block at the end, for a minimum of five blocks.

33. What value(s) of f satisfy $8f^2 + 2f - 4 = 0$?

This becomes $4f^2 + f - 2 = 0$, and the Quadratic Formula gives an answer of $f = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 4(-2)}}{2 \cdot 4} = \frac{-1 \pm \sqrt{33}}{8}$.

34. What is the sum of the 25 smallest positive odd numbers that are not perfect squares?

The first 25 odds are from 1 to 49, but include 1^2, 3^2, 5^2, and 7^2, so we really need to add up the $25 + 4 = 29$ smallest odd numbers, then subtract the four missing squares. The sum of the 29 smallest odd numbers is $29^2 = 841$, and the four squares add up to $1 + 9 + 25 + 49 = 84$, for an answer of $841 - 84 = 757$.

35. Set R is the set of positive multiples of five less than 69, Set S is the set of multiples of four from 40 to 444 inclusive, and Set T is the set of multiples of three greater than 33. How many elements are in the set $(T' \cap S) \cup R$?

R has 13 multiple of 5. S has 102 multiples of 4. $T' \cap S$ has all of S that is not in T, meaning all the elements of S that are not multiples of 3, which is $\frac{2}{3} \times 102 = \frac{204}{3} = 68$ elements. $(T' \cap S) \cup R$ adds in the 13 multiples of 5, but maybe not 40 and 60. 40 is already in $T' \cap S$, but not 60 (because it was a multiple of 3), so we only add 12 to 68, for an answer of 80.

36. What is the area of the convex pentagon with vertices at the points $(2, -3)$, $(-9, 5)$, $(8, 9)$, $(-6, -6)$, and $(-8, 9)$?

You could plot the points and determine the area of the pentagon by adding and subtracting a bunch of rectangles and right triangles. Alternatively, you can use the Shoelace Method. In clockwise order, the points are (8,9), $(2, -3)$, $(-6, -6)$, $(-9,5)$, $(-8,9)$, and (8.9) (you repeat the first line at the end in this algorithm). The left-hand products are $9 \cdot 2 = 18$, $-3(-6) = 18$, $-6(-9) = 54$, $5(-8) = -40$, and $9 \cdot 8 = 72$, for a left-hand total of 122. The right-hand products are $8(-3) = -24$, $2(-6) = -12$, $(-6)5 = -30$, $(-9)9 = -81$, and $(-8)9 = -72$, for a total of -219. Subtracting and dividing by two give $\frac{122-(-219)}{2} = \frac{341}{2}$.

37. Arrange the letters A-D in order of descending value.

$\boldsymbol{A = .9705}$ $\boldsymbol{B = \frac{2\pi}{7}}$ $\boldsymbol{C = \frac{\sqrt{2}}{2}}$ $\boldsymbol{D = \frac{8}{9}}$

$C \cong .707$, $D \cong .888$, and $B \cong \frac{6.28}{7} \cong .89^+$, for an answer of ABDC.

38. To allow his dog to run around in the park, its owner ties a rope between two trees that are 20 m apart so that there is 30 m of rope between the trees, and runs that rope through the dog's collar. What is the area, in square meters, that the dog can roam?

This is one definition of an ellipse: the locus of points the sum of whose distance from two fixed points (the foci) is a constant. This particular ellipse has a major axis of $20 + 5 + 5 = 30$, so a semi-major axis of 15. It has a semi-minor axis of $\sqrt{15^2 - 10^2} = 5\sqrt{3^2 - 2^2} = 5\sqrt{5}$, and thus an area of $\pi \cdot 15 \cdot 5\sqrt{5} = 75\pi\sqrt{5}$.

39. In a nine-element data set of integer test scores from 0 to 100 inclusive, the mean is 49, the median is 74, and the unique mode is 20. What is the largest possible value of the range?

The nine elements have a fixed sum of $9 \cdot 49 = 441$. When the elements are ordered, the central element is 74, and two of the lower elements are 20's. To get a large range, we'd like the lowest element to be very small (perhaps 0), and the highest element to be very large (perhaps 100). Because the mean is 49, which is much less than the median of 74, I suspect 100 isn't attainable, but 0 is, so I'll try a set that includes zero and the smallest other numbers I can, which would be the set 0, 1, 20, 20, 74, 75, 76, 77, x. The sum would be $343 + x = 441$, so that $x = 441 - 343 = 98$ is the largest possible value of the range.

40. Find a solution, in the form (h, j, k, m), of the system of equations $h + j + k + m = -10$, $hj + hk + hm + jk + jm + km = 23$, $hjk + hjm + hmk + jkm = 10$, $hjkm = -24$.

h, j, k, and m can be the roots of the polynomial $x^4 + 10x^3 + 23x^2 - 10x - 24 = 0$, which has easy-to-find roots of 1 and -1, and so factors to $(x - 1)(x + 1)(x^2 + 10x + 24) = 0$, then $(x - 1)(x + 1)(x + 6)(x + 4) = 0$, with roots of 1, -1, -6, and -4 in any order.