2016 Four-by-Four Competition
Solutions

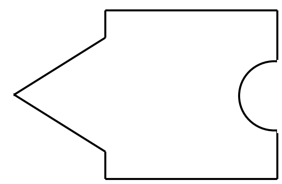
1. **When Tom guesses that Anne's number is 240, Anne grins because he's off by exactly 20% of her number. What is the sum of all possible values of Anne's number?**

   240 might be $80\% = \frac{4}{5}$ or $120\% = \frac{6}{5}$ of Anne's number, which might be $240 \cdot \frac{5}{4} = 60 \cdot 5 = 300$ or $240 \cdot \frac{5}{6} = 40 \cdot 5 = 200$, for an answer of $200 + 300 = 500$.

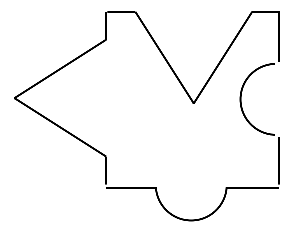
2. **When three fair six-sided dice are rolled, what is the probability that they show numbers that sum to 7?**

   A 7 could be made from 511 (three ways), 421 (six ways), 331 (three ways), or 322 (three ways), for a probability of $\frac{15}{216} = \frac{5}{72}$.

3. **The figure below is a square with two opposite sides modified by adding an equilateral triangle and removing a semi-circle. For your answer, draw a modified version of this figure (modifying the other two sides) that could tessellate a plane. Your drawing does not have to be to scale, but the shape must be clear to scorers.**



   So long as we cut a triangle from one of the sides and add a semi-circle to one of the sides, it doesn't matter which is which, and of course rotations are allowed. One possible shape is to the right.



4. **What is the missing term of the sequence 2, 3, 4, 9, 32, ___, 8896, …**

   The sequence itself isn't familiar; what about the differences? They are 1, 1, 5, 23, ?, and ?, which doesn't seem familiar, either. The numbers get big fast; perhaps there's some multiplying or exponentiation occurring; let's examine the prime factorizations… 2, 3, 2^2, 3^2, 2^5, ?, $8896 = 2^3 \cdot 1112 = 2^6 \cdot 139$. I note that there's a pattern of evens and odds, and the evens include 2, 2^2, 2^5, and 2^6; what if there's some multiplying of the term two previous? $2^2 = 2 \cdot 2$, $2^5 = 2^2 \cdot 2^5 = 2^2 \cdot 8$, and $2^6 \cdot 139 = 2^5 \cdot 278$. We notice that 2 is one less than 3 and 8 is one less than 9; might the pattern be multiply the term two previous by one less than the term one previous? $4 = 2(3-1)$, $9 = 3(4-1)$, and $32 = 4(9-1)$. If so, the answer should be $9(32-1) = 9 \cdot 31 = 279$. To check out answer, 8896 should be $32(279-1) = 32 \cdot 278$, which we already determined it was!

5. **Evaluate: $\mathbf{75060 \div 695}$**

   The standard algorithm gives 108.

6. **Evaluate: $\frac{\mathbf{10!}}{\mathbf{2^5 \cdot 5!}}$**

   $\frac{10!}{2^5 \cdot 5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2^5} = 5 \cdot 9 \cdot 7 \cdot 3 = 45 \cdot 21 = 945$

7. **A right circular cylinder has a lateral surface area that is half its total surface area, and it has a height of 12 m. What is its volume, in cubic meters?**

   The lateral surface area is $2\pi rh = 2\pi r \cdot 12 = 24\pi r$, which must be equal to the rest of the area, which is the top and bottom, giving $2\pi r^2 = 24\pi r$. This becomes $r = 12$, so that the volume is $\pi r^2 h = \pi \cdot 12^3 = 1728\pi$.

8. **What is the missing term of the harmonic sequence …, 168, 210, ___, 420, …?**

A harmonic sequence is the reciprocals of an arithmetic sequence, or can be thought of as some number divided by terms of an arithmetic sequence. In this case, 210 and 420 seems closely related, so we might think of them as $\frac{420}{2}$ and $\frac{420}{1}$, in which case the term between them would be $\frac{420}{1\frac{1}{2}} = \frac{420}{\frac{3}{2}} = 420 \cdot \frac{2}{3} = \frac{840}{3} = 280$.

9. **What is the perimeter, in meters, of a parallelogram with sides measuring 78 m and 34 m?**

A parallelogram has two pairs of congruent sides, so the perimeter will be $78 + 34 + 78 + 34 = 112 + 112 = 224$.

10. **A triangle is drawn with vertices at three of the endpoints of the axes of the ellipse with equation $\frac{(x-1)^2}{25} + \frac{(y+2)^2}{9} = 1$. What is the maximum possible area of such a triangle?**

The semi-axes of the ellipse have lengths of $\sqrt{25} = 5$ and $\sqrt{9} = 3$, so the triangle could have a base of $2 \cdot 5 = 10$ and a height of 3 for an area of $\frac{1}{2} \cdot 10 \cdot 3 = 5 \cdot 3 = 15$ or a base of $2 \cdot 3 = 6$ and a height of 5 for an area of $\frac{1}{2} \cdot 6 \cdot 5 = 3 \cdot 5 = 15$.

11. **The probability of rain is $\frac{2}{3}$, the probability that I watch TV is $\frac{1}{6}$, and the probability that I play board games is $\frac{3}{4}$. If these events are all independent, what is the probability that at most one of these things occurs?**

The probability that none of them occur is $\frac{1}{3} \cdot \frac{5}{6} \cdot \frac{1}{4} = \frac{5}{72}$, and the probabilities of each one occurring alone are $\frac{2}{3} \cdot \frac{5}{6} \cdot \frac{1}{4} = \frac{10}{72}$, $\frac{1}{3} \cdot \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{72}$, and $\frac{1}{3} \cdot \frac{5}{6} \cdot \frac{3}{4} = \frac{15}{72}$, for an answer of $\frac{5+10+1+15}{72} = \frac{31}{72}$.

12. **What is the shortest distance between the point $(85, 70)$ and the line $3x = 4y$?**

Rewriting the line as $3x - 4y = 0$, we can use the formula $\frac{|3 \cdot 85 - 4 \cdot 70|}{\sqrt{3^2 + (-4)^2}} = \frac{|255-280|}{5} = \frac{25}{5} = 5$.

13. **What is the volume, in cubic meters, of a right circular cone with a base radius of 6 m and a height of 8 m?**

For anything with a flat base and a pointy top, $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 6^2 \cdot 8 = \pi \cdot 2 \cdot 6 \cdot 8 = 12\pi \cdot 8 = 96\pi$.

14. **What number is half the sum of 8925 and the product of 85 and 79?**

$\frac{8925+85 \cdot 79}{2} = \frac{8925+6715}{2} = \frac{15640}{2} = 7{,}820$

15. **Express the binary numeral 11000100_2 in hexadecimal.**

Hexadecimal is base $16 = 2^4$, so every four binary digits is one digit in hexadecimal. $0100_2 = 4_{16}$ and $1100_2 = 8 + 4 = 12 = C_{16}$, for an answer of $C4_{16}$.

2016 Four-by-Four Competition
Solutions

16. My mini-Magic deck contains three different blue cards, four different green cards, and five different red cards. After I shuffle the deck, what is the probability that the cards are grouped by color?

There are 12! ways to arrange all the cards, and $3! \cdot 3! \cdot 4! \cdot 5!$ ways to arrange the cards grouped by color, for a probability of $\frac{3!\cdot 3!\cdot 4!\cdot 5!}{12!} = \frac{6\cdot 6\cdot 24}{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6} = \frac{1}{11\cdot 10\cdot 3\cdot 2\cdot 7} = \frac{1}{110\cdot 42} = \frac{1}{4620}$.

17. Sunit wishes to create a solution that is 40% sugar, but she only has 12 liters of 28% sugar solution and 18 liters of 64% sugar solution. What is the largest possible volume, in liters, of 40% sugar solution she can create by mixing these two?

40 is 12 more than 28 and 24 less than 64, so we'll need twice as much of the 28% solution as the 64% solution. Thus, we'll use 12 liters of the 28% solution and 6 liters of the 64% solution, for an answer of 18.

18. Quadrilateral ABCD has sides measuring 4 m, 6 m, 8 m, and 9 m. Quadrilateral EFGH is similar to ABCD, and has two sides measuring 8 m and 12 m. What is the smallest possible perimeter, in meters, of quadrilateral EFGH?

The 8 and 12 might correspond to the 4 & 6, with a distance ratio of 2, in which case the other two sides would be $2 \cdot 8 = 16$ and $2 \cdot 9 = 18$. But the 8 and 12 might correspond to the 6 and 9, with a distance ratio of $\frac{4}{3}$, in which case the other two sides would be $\frac{4}{3} \cdot 4 = \frac{16}{3}$ and $\frac{4}{3} \cdot 8 = \frac{32}{3}$, which are shorter and thus produce a smaller perimeter of $8 + 12 + \frac{16}{3} + \frac{32}{3} = 20 + \frac{48}{3} = 36$.

19. What is the sum of the positive two-digit odd integers?

You could do $50^2 - 5^2 = 2500 - 25 = 2475$, or use $\frac{45}{2}$ outer pairs with a sum of $99 + 11 = 110$ to get $\frac{45}{2} \cdot 110 = 45 \cdot 55 = 50^2 - 5^2 = 2500 - 25 = 2475$.

20. How many positive integers less than one million are palindromes written using at least one 7?

The palindromes could be of the forms A, AA, ABA, ABBA, ABCBA, or ABCCBA. There are 1 of each of the first two, $9 \cdot 10 - 8 \cdot 9 = 90 - 72 = 18$ of each of the next two, and $9 \cdot 10 \cdot 10 - 8 \cdot 9 \cdot 9 = 900 - 648 = 252$ of each of the last two, for a total of $504 + 36 + 2 = 542$.

21. Evaluate: $\frac{1}{2} \div \frac{1}{3} \times \frac{1}{4} - \frac{1}{6}\left(\frac{2}{3} \times \frac{3}{4}\right) \div \frac{5}{6}$

$\frac{1}{2} \div \frac{1}{3} \times \frac{1}{4} - \frac{1}{6}\left(\frac{2}{3} \times \frac{3}{4}\right) \div \frac{5}{6} = \frac{1}{2} \div \frac{1}{3} \times \frac{1}{4} - \frac{1}{6} \times \frac{1}{2} \div \frac{5}{6} = \frac{3}{8} - \frac{1}{10} = \frac{11}{40}$

22. Using the table to the right, evaluate
$\boldsymbol{j\left(p^{-1}(m(1))\right) \times k(p(2)) + m^{-1}\left(j(k(3))\right)}$

$j\left(p^{-1}(2)\right) \times k(3) + m^{-1}(j(-1))$
$= j(-2) \times (-1) + m^{-1}(1)$
$= (-1) \times (-1) + 2 = 1 + 2$
$= 3$

n	-2	-1	0	1	2	3
j(n)	-1	1	2	-2	0	3
k(n)	3	0	-2	1	2	-1
m(n)	0	-1	3	2	1	-2
p(n)	2	-2	-1	0	3	1

23. Every Platonic Solid has a "dual", which is a Platonic Solid formed by using the centers of each face of the original solid as the vertices of the new solid. What is the name of the Platonic Solid that is its own dual?

The five Platonic Solids are the Tetrahedron, Cube, Octahedron, Dodecahedron, and Icosahedron. The Tetrahedron has four faces and four vertices, so if we took the centers of each face we'd have four new vertices for another tetrahedron, so that's our answer. The Cube and Octahedron are duals of one another, as are the Dodecahedron and Icosahedron.

24. Evaluate: $\binom{22}{13}$

Cancelling is your friend. $\frac{22!}{13!\cdot 9!} = \frac{22\cdot 21\cdot 20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot 14}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} = 22\cdot 7\cdot 10\cdot 19\cdot 17 = 323\cdot 1540 = 497{,}420$

25. What is the area, in square meters, of a triangle with sides measuring 5 m, 7 m, and 8 m?

Stewart's Theorem uses the semi-perimeter $s = \frac{5+7+8}{2} = \frac{20}{2} = 10$: $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{10\cdot 5\cdot 3\cdot 2} = 10\sqrt{3}$.

26. What is the inverse of the matrix $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$**?**

This is easiest if you remember that the inverse is the transposed matrix of cofactors divided by the determinant. Our determinant is $3\cdot 5 - 2\cdot 7 = 15 - 14 = 1$, so we just need the transposed matrix of cofactors. The matrix of cofactors is $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$, so the answer is $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$.

27. James starts at 695 and colors every 8,040th number on the number line brown. Danielle starts at 5,489 and colors every 9,375th number on the number line white. What is the smallest possible positive difference between a brown number and a white number?

Their numbers are initially $5489 - 695 = 4794$ apart, but that can change by 8040 or 9375 or any combination of the two. $8040 = 8\cdot 1005 = 2^3\cdot 5\cdot 201 = 2^3\cdot 3\cdot 5\cdot 67$ and $9375 = 25\cdot 375 = 5^4\cdot 15 = 5^5\cdot 3$, so their GCF is $3\cdot 5 = 15$. This means that no matter how we change our use of 8040's and 9375's, we'll always alter the 4794 difference by a multiple of 15, which means that the smallest possible difference will be the remainder when 4794 is divided by 15, which is the same as 294's remainder, which is the same as -6's remainder, so our answer is 6.

28. What is the value of w in the solution to the system of equations $t + u + v + w = 48$, $2t - 3u + 2v - 2w = 82$, and $t - 2u + v - w = 56$?

Subtracting the last from the first gives $3u + 2w = -8$, and subtracting the second from twice the first gives $5u + 4w = 14$. Subtracting the second of these from twice the first of these gives $u = -30$. Substituting this into the first of the two gives $-90 + 2w = -8$, then $2w = 82$, yielding $w = 41$.

2016 Four-by-Four Competition
Solutions

29. What is the product of the six sixth-roots of 64?

The six roots all satisfy $x^6 = 64$, which can be written $x^6 - 64 = 0$, the product of the roots of which is $\frac{-64}{1} = -64$.

30. Evaluate: $\int_2^5 \frac{1}{n^2+n} dn$

$\int_2^5 \frac{1}{n^2+n} dn = \int_2^5 \left(\frac{1}{n} - \frac{1}{n+1}\right) dn = \ln n - \ln(n+1)|_2^5 = \ln\left(\frac{n}{n+1}\right)\Big|_2^5 = \ln\frac{5}{6} - \ln\frac{2}{3} =$
$\ln\left(\frac{5}{6} \cdot \frac{3}{2}\right) = \ln\frac{5}{4}$

31. What is the sum of all sums that are more likely when three fair six-sided dice are rolled than they are when two dice are rolled? E.g. 18 has a higher probability when three dice are rolled than when two dice are rolled, so it will be part of your sum.

Obviously, 18 through 13 will be part of the sum. 2 & 12 have low two-dice probabilities of $\frac{1}{36} = \frac{6}{216}$, so are good candidates, but there's no way to roll a 2 with three dice, and similarly other low numbers will NOT be good candidates. With three dice, you can roll a 12 as 651 (six ways), 642 (six ways), oh, we're already better than with two dice. 11 has a two-dice probability of $\frac{2}{36} = \frac{12}{216}$, and on three dice can be rolled as 641 (six ways), 632 (six ways), 551 (three ways), and it's already better. 10 has a two-dice probability of $\frac{3}{36} = \frac{18}{216}$, and on three dice can be rolled as 631 (six ways), 622 (three ways), 541 (six ways), 532 (six ways), and it's already better. 9 ha a two-dice probability of $\frac{4}{36} = \frac{24}{216}$, and on three dice can be rolled as 621 (six ways), 531 (six ways), 522 (three ways), 441 (three ways), 432 (six ways), and 333 (one way), just barely better. So our answer will be the sum of the numbers from 9 to 18, which is 10 numbers, so five outer pairs that sum to 27, for a total of 135.

32. My piggy bank contains 26 coins, each of which is either a penny, nickel, dime, or quarter. The total value of the coins is \$2.94, and there are twice as many of the coins with the smallest radius as there are of the coins with the largest radius. How many nickels are there?

There must be at least 4 pennies, and perhaps there are 9, 14, etc. So we're now looking for 22 coins that total to \$2.90, so the average value is a little over 10 cents. The coin with the smallest radius is a dime, and that with the largest radius is a quarter, so we might have 2&1, 4&2, 6&3, 8&4, 10&5, 12&6, or 14&7 dimes and quarters respectively. To get an average value over 10 cents, we don't want very many pennies or nickels, so focus on the larger pairs. 14 dimes & 7 quarters are worth $\$1.40 + \$1.75$, which is too much. 12 & 6 give $\$1.20 + \$1.50 = 2.70$, leaving $\$.20$ left for four coins, which would be the desired nickels.

33. Evaluate: $\cot\frac{8369\pi}{6}$

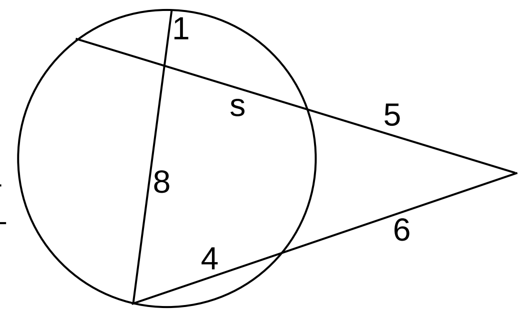
$\cot\frac{8369\pi}{6} = \cot\frac{2369\pi}{6} = \cot\frac{569\pi}{6} = \cot\frac{29\pi}{6} = \cot\frac{5\pi}{6} = \frac{1}{\tan\frac{5\pi}{6}} = \frac{1}{-\frac{1}{\sqrt{3}}} = -\sqrt{3}$

2016 Four-by-Four Competition
Solutions

34. Express in simplest radical form: $\sqrt{66825}$

$\sqrt{66825} = 5\sqrt{2673} = 5 \cdot 3\sqrt{297} = 5 \cdot 3 \cdot 3\sqrt{33} = 45\sqrt{33}$

35. The figure to the right shows a circle with two secants and a chord, with most segment lengths labeled in meters. What is the smallest possible value of s?



The two secants allow us to write $6(6+4) = 5(5+x)$, which becomes $60 = 5(5+x)$, giving $12 = 5 + x$ and finally $x = 7$. The two chords allow us to write $8 \cdot 1 = 8 = s(7-s) = 7s - s^2$, giving $s^2 - 7s + 8 = 0$ for an answer of $s = \frac{7 \pm \sqrt{49-32}}{2} = \frac{7 \pm \sqrt{17}}{2}$, with the minus giving the smaller value.

36. Set Q is the set of all multiples of 7 ending in 1, and Set R is the set of all positive three-digit multiples of 3. How many elements are in the set $R \cap Q'$?

There are 900 three-digit numbers, so set R has $900 \div 3 = 300$ elements. The desired anwer needs to exclude any of those which are also elements of Q, meaning anything that is a three-digit multiple of 3 and 7 that ends with 1. These would be $7 \cdot 3 \cdot 11$, $7 \cdot 3 \cdot 21$, $7 \cdot 3 \cdot 31$, and $7 \cdot 3 \cdot 41$, which is four elements, making our answer $300 - 4 = 296$.

37. Consider the function $s(t) = 2t^3 - 3t + 4$ on the interval $[1, 5]$. What is the value of t on this interval that satisfies the Mean Value Theorem?

The MVT says that the "average derivative" over an interval actually occurs somewhere in that interval. The endpoints of the curve are $(1,3)$ and $(5,239)$, so the "average derivative" is $\frac{239-3}{5-1} = \frac{236}{4} = 59$. The actual derivative at the desired point is $6t^2 - 3 = 59$ which becomes $6t^2 = 62$, then $t^2 = \frac{31}{3}$, and finally $t = \sqrt{\frac{31}{3}} = \frac{\sqrt{93}}{3}$.

38. What are the coordinates, in the form (x, y) of the x-intercept of the line through the points $(532, 902)$ and $(182, -123)$?

This line has a slope of $\frac{902-(-123)}{532-182} = \frac{1025}{350} = \frac{41}{14}$, so y changes by 41 whenever x changes by 14. From $(182, -123)$ to $(?, 0)$, we'll need y to increase by 41 three times, so x will increase by 14 three times to give $182 + 14 \cdot 3 = 182 + 42 = 224$ for an answer of $(224,0)$.

39. What is the sum of the positive three-digit multiples of four that contain at least one 6?

It may be easiest to count everything of the form 6xx, then the x6x's without a leading 6, then the xx6's with no other 6's. The first group is from 600 to 696, which is 25 numbers, so $\frac{25}{2}$ outer pairs that sum to 1296, for a total of $\frac{25}{2} \cdot 1296 = 25 \cdot 648 = 16{,}200$. The x6x's can start with 8 different numbers (1-9 minus 6) and end with 0, 4, or 8, for a total of $8 \cdot 3 = 24$ such numbers. Their hundred digits will total to $3900 \cdot 3 = 11{,}700$, their tens digits will total $60 \cdot 24 = 1{,}440$, and their units digits will total $12 \cdot 8 = 96$, for a total contribution of 13,236. The xx6's can start with the same 8 digits and have 1,3, 5, 7, or 9 in the middle, for a total of $8 \cdot 5 = 40$ such numbers. Their hundreds digits will total $3900 \times 5 = 19{,}500$, their tens digits will total $250 \cdot 8 = 2{,}000$, and their units digits will total $6 \cdot 40 = 240$, for a total contribution of 21,740 and an answer of 51,176.

40. In a triangle with sides measuring 4 m, 6 m, and 9 m, a cevian of length 5 is drawn to the longest side. What is the length of the longer segment into which it divides that side?

Using Stewart's Theorem, we can write $36x + 16(9 - x) = 225 + 9x(9 - x)$, where x is our answer. This becomes $20x + 144 = 225 + 81x - 9x^2$, then $9x^2 - 61x - 81 = 0$, with roots of $x = \frac{61 \pm \sqrt{61^2 - 4 \cdot 9(-81)}}{2 \cdot 9} = \frac{61 \pm \sqrt{3721 + 2916}}{18} = \frac{61 \pm \sqrt{6637}}{18}$, with the plus giving the needed positive root.