

# 2014 Four-by-Four Competition Solutions

1. **What are the coordinates, in the form  $(x, y)$ , of the vertex of the parabola  $x = 2y^2 - 24y + 89$ ?**

Note that  $x$  &  $y$  are in non-standard places, so the axis of symmetry is  $y = -\frac{b}{2a} = -\frac{-24}{2 \times 2} = \frac{24}{4} = 6$  and the  $x$ -coordinate of the vertex is  $x = 2 \times 6^2 - 24 \times 6 + 89 = 72 - 144 + 89 = -72 + 89 = 17$ , for an answer of  $(17, 6)$ .

2. **What is the area, in square meters, of a right triangle with a  $30^\circ$  angle and a hypotenuse measuring 24 m?**

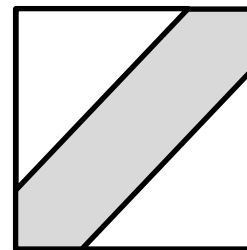
It's a 30-60-90 triangle, so the short leg is half the hypotenuse (12), and the long leg is  $\sqrt{3}$  times that ( $12\sqrt{3}$ ), for an area of  $\frac{1}{2} \times 12 \times 12\sqrt{3} = 72\sqrt{3}$ .

3. **Evaluate:  $(2i)^3 - (4i)^5 + (7i)^2 - (9i - 8)$**

$$-8i - 1024i - 49 - 9i + 8 = -1041i - 41$$

4. **When two points are chosen on a yardstick, what is the probability that they are within six inches of one another?**

The easiest way to think of problems like this is geometrically in two dimensions as shown in the figure to the right. The horizontal dimension represents the position of the "first" point along the yardstick from 0 to 36, and the vertical dimension represents the position of the "second" point along the yardstick from 0 to 36. The shaded region in the middle represents cases where the two points are within 6 inches of one another, and we'll find its area by subtracting the two isosceles right triangles, which are equivalent to a square. The



probability is thus  $\frac{36^2 - 30^2}{36^2} = \frac{6^2 - 5^2}{6^2} = \frac{36 - 25}{36} = \frac{11}{36}$ .

5. **What is the sum of the odd integers between 6 and 24?**

The sum of the 12 smallest positive odd numbers is  $12^2 = 144$ , and the sum of the three smallest is  $3^2 = 9$ , for an answer of  $144 - 9 = 135$ .

6. **An angle's supplement is ten times its complement. What is the angle's measure, in degrees?**

We can write  $180 - x = 10(90 - x) = 900 - 10x$ , then  $9x = 720$ , giving  $x = 80$ .

7. **What number is 87 less than two-thirds of the product of 327 and 18?**

$$\frac{2}{3} \times 18 \times 327 - 87 = 12 \times 327 - 87 = 3837$$

8. **What are the coordinates, in the form  $(x, y)$ , of the center of the hyperbola with equation  $4x^2 - 6y^2 + 8x + 10y = 12$ ?**

Completing the squares gives  $4(x + 1)^2 - 6\left(y - \frac{5}{6}\right)^2 = \dots$ , for an answer of  $\left(-1, \frac{5}{6}\right)$ .

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9. Evaluate:  $\left(\frac{81}{16}\right)^{-\frac{3}{2}}$

$$\left(\frac{81}{16}\right)^{-\frac{3}{2}} = \left(\frac{16}{81}\right)^{\frac{3}{2}} = \left(\frac{4}{9}\right)^3 = \frac{64}{729}$$

10. What value(s) of  $b$  satisfy  $9(2b + 7) = 3(8b - 5)$ ?

$18b + 63 = 24b - 15$  becomes  $78 = 6b$ , giving  $b = 13$ .

11. In a regular polygon, the sum of the interior angles is 84 times the measure of an exterior angle. How many sides does the polygon have?

$180(n - 2) = 84 \times \frac{360}{n}$  becomes  $n(n - 2) = 168$ . Because  $169 = 13^2$ , we can see that  $n = 14$  is the answer.

12. In a seven-element set of integer test scores from 0 to 100 inclusive, the mean is 79, the unique mode is 68, and the range is 45. What is the largest possible value of the median?

Writing the elements in order gives  $x, 68, 68, y, \_, \_, x + 45$ . The seven elements must add up to  $7 \times 79 = 553$ , and to get a large median we'd like the other elements to be as small as they can, so we can write  $y - 42, 68, 68, y, y + 1, y + 2, y + 3$ . These elements sum to  $5y + 100 = 553$ , which becomes  $5y = 453$ , giving  $y = 90.6$ , for an answer of 90.

13. Evaluate:  $987.6 + 53.19 + 8.529 + 9810$

Keeping the decimal points aligned, we get a sum of 1050.3.

14. What is the area in the Cartesian Plane enclosed by the locus of points satisfying  $5x^2 + 6y^2 = 7$ ?

Writing the ellipse in standard form, we get  $\frac{x^2}{\frac{7}{5}} + \frac{y^2}{\frac{7}{6}} = 1$ , with semi-axes of  $\sqrt{\frac{7}{5}}$  and  $\sqrt{\frac{7}{6}}$ , for an area of  $\pi ab = \pi \sqrt{\frac{7}{5}} \sqrt{\frac{7}{6}} = \frac{7\pi}{\sqrt{30}} = \frac{7\pi\sqrt{30}}{30}$ .

15. A llama is tethered to an external corner of a hexagonal grain silo with no openings at ground level. If the hexagon has sides measuring 6 m and the llama leash is 15 m long, what is the area, in square meters, of the region in which the llama can roam?

The llama can roam in  $\frac{2}{3}$  of a circle with a radius of 15,  $\frac{1}{3}$  of a circle with a radius of 9, and  $\frac{1}{3}$  of a circle with a radius of 3, for an answer of  $\frac{2}{3} \times 15^2\pi + \frac{1}{3} \times 9^2\pi + \frac{1}{3} \times 3^2\pi = 180\pi$ .

16. What is the sum of the positive integers that are factors of 420?

$420 = 2^2 \times 3 \times 5 \times 7$ , so the sum of its factors is  $(1 + 2 + 4)(1 + 3)(1 + 5)(1 + 7) = 7 \times 4 \times 6 \times 8 = 4 \times 336 = 1344$ .

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**17. What is the period of  $y = |2 \sin(3x) - 4 \cos(5x)|$ ?**

The sub-functions have periods of  $\frac{2\pi}{3}$  and  $\frac{2\pi}{5}$ , for a combined period of  $2\pi$ . The absolute value might cause the period to be divided by an integer, often 2 for trigonometric functions, but it requires a little thought to figure out if that is the case. A rough graph often helps, or you can consider substituting  $x + \pi$  into the function or sub-functions. In any event, for this function the absolute value causes the period to be divided by 2, giving a period of  $\pi$ .

**18. When the magic number is increased by 357, this result is divided by 7, and this result is decreased by 197, the final result is 247. What is the magic number?**

Working backwards,  $247 + 197 = 444$ , then  $444 \times 7 = 3108$ , and  $3108 - 357 = 2751$ .

**19. A  $135^\circ$  sector of a circle has an area of  $294\pi$  m<sup>2</sup>. What is its perimeter, in meters?**

$\frac{3}{8}\pi r^2 = 294\pi$  becomes  $r^2 = 294 \times \frac{8}{3} = 8 \times 98 = 16 \times 49$ , giving  $r = 4 \times 7 = 28$ , for a perimeter of  $28 + 28 + \frac{3}{8} \times 2 \times 28\pi = 56 + 21\pi$ .

**20. When four cards are drawn from a standard 52-card deck, what is the probability that they are four different suits and four different ranks?**

The first card is irrelevant, after which there are three suits of 12 good cards, then two suits of 11 good cards, and finally one suit of 10 good cards, for a probability of  $\frac{36}{51} \times \frac{22}{50} \times \frac{10}{49} = \frac{12 \times 22}{17 \times 5 \times 49} = \frac{264}{4165}$ .

**21. Evaluate, expressing your answer in standard form:  $\frac{(8.4 \times 10^3)(3.5 \times 10^{-1})}{(2.8 \times 10^{-2})(1.5 \times 10^0)}$**

$$\frac{84 \times 35}{28 \times 15} \times 10^4 = \frac{3 \times 7}{3} \times 10^4 = 70,000$$

**22. How many positive five-digit integers are palindromic multiples of four?**

The numbers must be of the form ABCBA, and BA must be a multiple of four. There are 25 such multiples of 4, but we can't use 00, 20, 40, 60, or 80, so there are really only 20 choices for BA. Combined with 10 choices for C, the answer is  $10 \times 20 = 200$ .

**23. What value(s) of  $c$  satisfy  $24c^2 - 49c - 108 = 0$ ?**

We'd prefer not to plug these numbers into the Quadratic Equation, so let's see if we can factor it. Although the coefficients are large, they do not have any collective common factor, so we can be sure that neither of the two factors we seek contains coefficients with a common factor. In addition, the central term is not a multiple of 2 or 3, so neither 24 nor 108 can be factored into two multiples of 2 or 3. 24 can be factored to 24 & 1, 12 & 2, 8 & 3, or 6 & 4, , but only 24 & 1 and 8 & 3 might work. 108 can be factored to 108 & 1, 54 & 2, 36 & 3, 27 & 4, 18 & 6, or 12 & 9, but only 108 & 1 and 27 & 4 might work. Of these options,  $(8c - 27)(3c + 4) = 0$  works, giving roots of  $\frac{27}{8}$  and  $-\frac{4}{3}$ .

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- 24. Set W is the set of positive three-digit integers containing three different digits, Set V is the set of positive three-digit integers with at least one digit that is a 3, and Set T is the set of positive three-digit integers that are multiples of three. How many elements are in the set  $W \cap (V' \cup T)$ ?**

We're looking for numbers with three different digits AND either don't have 3 as a digit or are a multiple of 3 (or both). It's easy to figure out how many have three different digits none of which are a 3; there are 8 choices for the first digit (just not 3), 8 for the next (not 3 or the first digit, but 0 is okay), and 7 for the third, for a total of  $8 \times 8 \times 7 = 64 \times 7 = 448$  numbers. In addition to these 448 numbers, we need to add numbers with three different digits, one of which IS a 3, and which are multiples of 3. The other two digits could be 06, 09, 12, 15, 18, 24, 27, 45, 48, 57, 69, or 78. Combined with the digit 3, each of these sets of digits could be arranged in 6 ways, for a total of  $6 \times 12 = 72$  new numbers. However, the sets 306 and 309 collectively contain 4 invalid numbers: 036, 063, 039, and 093, meaning this case only adds  $72 - 4 = 68$  new numbers, for an answer of  $448 + 68 = 516$ .

- 25. What are the coordinates, in the form  $(x, y)$ , of the local maximum of the function  $y = x^3 - 12x - 9$ ?**

The maximum will occur where  $y' = 3x^2 - 12 = 0$ , which is  $x = \pm 2$ . Obviously, the function gets very positive for large  $x$  and very negative for small  $x$ , so the local maximum will be at  $x = -2$ , giving  $y = -8 + 24 - 9 = 24 - 17 = 7$ , for an answer of  $(-2, 7)$ .

- 26. A shepherd has his flock at  $(5, 7)$  and wishes to take it to the river ( $y = x - 20$ ) before going to the ranch at  $(0, 0)$ . What is the shortest total distance they can travel?**

If we reflect the ranch across the river to the point  $(20, -20)$ , the straight line will be the shortest distance:  $\sqrt{15^2 + 27^2} = 3\sqrt{5^2 + 9^2} = 3\sqrt{25 + 81} = 3\sqrt{106}$ .

- 27. How many elements of a geometric sequence with first term 3 and common ratio 2 are also part of a harmonic sequence with first term 1200 and second term 1000?**

1000 is  $\frac{5}{6}$  of 1200, so the harmonic sequence is 6000 divided by consecutive integers starting with 5.  $6000 = 2^4 \times 3 \times 5^3$ , so the numbers 3, 6, 12, 24, and 48 can eventually be generated by the harmonic sequence, and are the starting terms of the geometric sequence, so the answer is 5.

- 28. A right rectangular prism has a length, width, and depth that are all integers when measured in meters. If the surface area of the prism is  $432 \text{ m}^2$ , what is the maximum possible volume of the prism, in cubic meters?**

We can write  $2ab + 2bc + 2ac = 432$ , which becomes  $ab + bc + ac = 216$ . For a maximum volume, we'd like  $a$ ,  $b$ , and  $c$  to be as close to one another as possible, so we'd like  $ab = 72$ , so that each dimension is near 8.5. It's possible that all the dimensions would be even, in which case we could write  $de + df + ef = d(e + f) + ef = 54$ . Guessing values around 4.25 quickly shows that  $e$  and  $f$  can be 4 and 6, with  $d$  being 3, for dimensions of 6, 8, and 12 and a volume of  $6 \times 8 \times 12 = 6 \times 96 = 600 - 24 = 576$ .

- 29. Evaluate:  $1098^2 - 912^2$**

$$1098^2 - 912^2 = (1098 + 912)(1098 - 912) = 2010 \times 186 = 373,860$$

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**30. What is the remainder when  $11^{111}$  is divided by 1000?**

$(1 + 10)^{111} = 1^{111} + 111 \times 1^{110} \times 10 + 111 \times 55 \times 1^{109} \times 10^2 + \dots$ , with the last and all subsequent terms not influencing our answer, so that we get  $1 + 110 + 500 = 611$ .

**31. My piggy bank contains 29 coins, each of which is either a penny, nickel, dime, or quarter. The total value of all the coins is \$2.86, and the number of quarters is equal to the sum of the numbers of nickels and dimes, and is less than the number of pennies. How many nickels are there?**

The number of pennies could be 1, 6, 11, 16, 21, or 26, because you need at least five pennies to adjust for changes in the other coins. 21 and 26 can't possibly give a total of \$2.86. 1, 6, 11, and 16 leave 28, 23, 18, and 13 other coins, half of which must be quarters, so really there must be 1 or 11 pennies with 14 or 9 quarters respectively. The former does not have fewer quarters than pennies, so there are 11 pennies and 9 quarters, for a subtotal of \$2.36, leaving \$0.50 for 9 coins, which must be 8 nickels and 1 dime.

**32. When 123 people were surveyed, 94 liked sunny days, and 75 liked windy days. What is the smallest number of people who could have liked both?**

There are  $123 - 94 = 29$  people who don't like sunny days. If they all liked windy days, that would mean that  $75 - 29 = 46$  liked both, which is the maximum.

**33. A Hailstone Sequence is a recursive sequence of positive integers in which  $a_n = \frac{a_{n-1}}{2}$  if  $a_n$  is even, but  $a_n = 3a_{n-1} + 1$  if  $a_n$  is odd. Choosing different values of  $a_1$  results in different Hailstone Sequences. What is the smallest value of  $a_1$  that will generate 11 as part of its Hailstone Sequence?**

Starting with 1 gives 4, then 2, then 1, then repeats. 2 is the same story. Starting with 3 gives 10, 5, 16, 8, 4, 2, 1, then repeats. 4 repeats quickly. Starting with 5 gives 16, 8, uh-oh. Starting with 6 gives 3, then we've been here before. Starting with 7 gives 22, 11, yay!

**34. A convex polygon can have over 1000 diagonals drawn in it. What is the smallest number of sides the polygon could have?**

$\frac{n(n-3)}{2} > 1000$ , so  $n(n-3) > 2000$ . I know  $45^2 = 2025$ , so trying 46 might work.  
 $46 \times 43 = 138 + 1840 = 1978$ , so 47 will be the answer.

**35. What is the equation (in the form  $Ax + By + Cz = D$  where  $A$  is positive and  $A, B, C$ , and  $D$  have a collective greatest common factor of 1) of the plane containing the point  $(-2, -6, -3)$  and the line  $\frac{x-4}{8} = \frac{y+5}{8} = \frac{z-1}{2}$ ?**

The line contains the point  $(4, -5, 1)$  and runs in the direction  $\langle 8, 8, 2 \rangle$  which is the same as  $\langle 4, 4, 1 \rangle$ . The vector from  $(-2, -6, -3)$  to  $(4, -5, 1)$  is  $\langle 6, 1, 4 \rangle$ , and both this and  $\langle 4, 4, 1 \rangle$  lie in our plane. Their cross product is  $\langle -15, 10, 20 \rangle$ , which is in the same direction as  $\langle -3, 2, 4 \rangle$  and is perpendicular to our plane, so that its equation is  $3x - 2y - 4z = d$ . Substituting the point  $(4, -5, 1)$  gives  $d = 3 \times 4 - 2(-5) - 4 \times 1 = 12 + 10 - 4 = 18$  for an answer of  $3x - 2y - 4z = 18$ .

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- 36. The members of the investment club contributed evenly to buy BrownCo. If there had been one more member, each would have contributed three billion fewer dollars. If there had been four fewer members, each would have contributed fifteen billion more dollars. What was the purchase price of BrownCo, in trillions of dollars as a decimal?**

If  $N$  is the number of investors and  $p$  is the amount they each paid, we can write  $Np = (N + 1)(p - 3) = (N - 4)(p + 15)$ , which gives the equations  $3 = p - 3N$  and  $60 = -4p + 15N$ . Adding five of the former to the latter gives  $75 = p$ , so that  $N = 24$ , and the sale price of BrownCo was  $24 \times 75 = 1800$  billion dollars, which is 1.8 trillion dollars.

- 37. Ten lines are drawn in a plane. When considering all 45 possible pairs of lines, there are at least two pairs in which the lines are parallel, at least two pairs in which the lines are perpendicular, and at least two pairs in which the lines meet at an angle other than ninety degrees. What is the fewest number of regions into which the ten lines can divide the plane?**

The more parallel lines there are, the fewer regions there will be. There can be 8 parallel lines, one perpendicular line, and one line at another angle. If that tenth line passes through a point of perpendicularity, each of the 9 regions made by the parallel lines is divided into three regions, for a minimum of 27 regions.

- 38. A triangle has angles measuring  $30^\circ$  and  $45^\circ$  and a side measuring 18 m. What is the shortest possible length, in meters, of another side of the triangle?**

The triangle is 30-45-105. The shortest side is opposite the  $30^\circ$  angle, and this side will be smallest in the smallest triangle with a side of 18, which is one where 18 is the longest side (opposite the  $105^\circ$  angle). By the Law of Sines, we get  $\frac{x}{\sin 30^\circ} = \frac{18}{\sin 105^\circ}$ , which becomes

$$\frac{x}{\frac{1}{2}} = \frac{18}{\frac{\sqrt{6}+\sqrt{2}}{4}}, \text{ then } x = \frac{36}{\sqrt{6}+\sqrt{2}} = \frac{36(\sqrt{6}-\sqrt{2})}{6-2} = 9\sqrt{6} - 9\sqrt{2}.$$

- 39. When six friends sit in six seats in a row at the movies, Reva and Quincy sit next to one another, Pam and Oly do not, Nora sits somewhere to Mark's left, and Reva sits somewhere to Pam's right. If Quincy sits on one end of the row and Oly does not, how many seating arrangements are possible?**

Reva, Quincy, & Pam combine to force Quincy to the right end and Reva next to him. Putting Nora to Mark's left in the other four seats gives six options, but three of them would require Pam & Oly to sit next to one another. Of the remaining three, two have an empty seat on the left end which must be Pam, and the other allows two possible solutions, for an answer of 4 possible arrangements.

- 40. What is the value of  $g$  in the solution to the system of equations  $d + f + g + h + j = 11$ ,  $d - 2f - 3g + 3h - 2j = -49$ ,  $2d + f - h + j = 15$ , and  $2d - f + g - 3h - j = 68$ ?**

Eliminating  $f$  using the first equation gives  $3d - g + 5h = -27$ ,  $d - g - 2h = 4$ , and  $3d + 2g - 2h = 79$ . Eliminating  $d$  using the first equation gives  $2g + 11h = -39$  and  $-3g + 7h = -106$ . Adding three of the first equation to two of the second gives  $47h = -329$ , so that  $h = -7$  and thus  $g = 19$ .