2017 Ciphering Time Trials
Solutions

1. **What are the coordinates, in the form (x, y), of the midpoint of the line segment connecting the points $(7, 4)$ and $(6, -7)$?**

   The midpoint will be $\left(\frac{7+6}{2}, \frac{4+(-7)}{2}\right)$, which is $\left(\frac{13}{2}, -\frac{3}{2}\right)$.

2. **If the cosine of an angle in the third quadrant is $-\frac{1}{4}$, what is the sine of that angle?**

   In the third quadrant, the sine will also be negative, for an answer of $-\sqrt{1-\left(\frac{1}{4}\right)^2} = -\sqrt{1-\frac{1}{16}} = -\sqrt{\frac{15}{16}} = -\frac{\sqrt{15}}{4}$.

3. **What is the sum of the distinct prime factors of 675?**

   $675 = 5^2 \cdot 27 = 3^3 \cdot 5^2$, for an answer of $3 + 5 = 8$.

4. **What is the value of the discriminant of $4g^2 + 9g + 2 = 0$?**

   The discriminant is $b^2 - 4ac = 9^2 - 4 \cdot 4 \cdot 2 = 81 - 32 = 49$.

5. **What is the equation of the plane through the point $(9, 5, 9)$ and perpendicular to the vector $< 5, 7, 4 >$? Express your answer in the form $Ax + By + Cz = D$, where $A > 0$ and A, B, and C are collectively relatively prime.**

   Planes look a lot like their perpendicular vectors, so this one will be $5x + 7y + 4z = D$, where substituting gives $5 \cdot 9 + 7 \cdot 5 + 4 \cdot 9 = D = 116$, for an answer of $5x + 7y + 4z = 116$.

6. **What is the value, as a decimal number of dollars, of 40 quarters, 4 dimes, 83 nickels, and 587 pennies?**

   The coins are worth \$10.00, \$0.40, \$4.15, and \$5.87, for a total of \$20.42.

7. **If $g(h) = -8h + 5 - \frac{5}{h}$, evaluate $g(3)$.**

   $g(3) = -8(3) + 5 - \frac{5}{3} = -24 + 5 - \frac{5}{3} = -19 - \frac{5}{3} = -\frac{57}{3} - \frac{5}{3} = -\frac{62}{3}$

8. **What is the next term of a harmonic sequence beginning with 16 and 6?**

   A harmonic sequence is a constant divided by the terms of an arithmetic sequence. This harmonic sequence can be considered $6 \cdot 16$ divided by 6, then 16, then 26… for an answer of $\frac{96}{26} = \frac{48}{13}$.

9. **A cylinder with a base radius of 6 m and a height of 4 m is inscribed in a sphere. What is the volume, in cubic meters, of the sphere?**

   The sphere has a diameter of $\sqrt{12^2 + 4^2} = 4\sqrt{3^2 + 1^2} = 4\sqrt{10}$, so its radius is $2\sqrt{10}$, for a volume of $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(2\sqrt{10}\right)^3 = \frac{4}{3}\pi \cdot 8 \cdot 10\sqrt{10} = \frac{320\pi\sqrt{10}}{3}$.

10. **What is the product of the roots of $-3b^3 - 4b^2 + 9b + 7 = 0$?**

    The product of the roots of a polynomial is $(-1)^n \frac{z}{a} = (-1)^3 \cdot \frac{7}{-3} = -1 \cdot \frac{7}{-3} = \frac{7}{3}$.

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11. How many diagonals can be drawn in a regular 32-gon, not counting the shortest diagonals?

Each vertex can have a diagonal to any other vertex that is not itself or its nearest neighbor, so you might think there are $32 \cdot 29$ diagonals. However, this method counts each diagonal twice (once from each vertex), for so there are really $\frac{32 \cdot 29}{2} = 16 \cdot 29 = 480 - 16 = 464$ diagonals. Of these, the 32 that skip just one vertex are the shortest, for an answer of $464 - 32 = 432$.

12. The probabilities that Dharma, Eugene, and Fernbelly pass tomorrow's Algebra test are $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{1}{6}$, respectively. If these events are independent, what is the probability that exactly two of the three pass the test?

The answer will be $\frac{3}{4} \cdot \frac{2}{5} \cdot \frac{5}{6} + \frac{3}{4} \cdot \frac{3}{5} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{2}{5} \cdot \frac{1}{6} = \frac{30+9+2}{120} = \frac{41}{120}$.

13. What is the measure, in degrees, of an interior angle of a regular 24-gon?

The "exterior" angle of a 24-gon is $\frac{360}{24} = \frac{30}{2} = 15$ so the interior angle will be $180 - 15 = 165$.

14. Set D is the set of positive two-digit even numbers, and Set E is the set of multiples of 10 less than 1000. What is the number of elements in the set $D \cap E$?

We're looking for multiples of 10 less than 1000 that are also two-digit even numbers. All multiples of 10 are even, and all two-digit numbers are less than 1000, so we're looking for the two-digit multiples of 10, of which there are 9.

15. What value(s) of b satisfy $8(6b - 6) = 4(7b + 8) + 576$?

This becomes $48b - 48 = 28b + 32 + 576 = 28b + 608$, then $20b = 656$, for an answer of $b = \frac{656}{20} = \frac{164}{5}$.

16. Evaluate: $875 + 7130$

The standard algorithm gives 8005.

17. What is the volume, in cubic meters, of a rectangular pyramid with base edges measuring 6 m and 5 m and a height of 9 m?

$V = \frac{1}{3}Bh = \frac{1}{3}wdh = \frac{6 \cdot 5 \cdot 9}{3} = 6 \cdot 5 \cdot 3 = 90$

18. Evaluate: $\sum_{c=3}^{11} \frac{1}{c^2+6c+8}$

$\frac{1}{c^2+6c+8} = \frac{1}{(c+2)(c+4)} = \frac{1}{2}\left(\frac{1}{c+2} - \frac{1}{c+4}\right)$, so the series becomes $\frac{1}{2}\left(\left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \cdots + \left(\frac{1}{13} - \frac{1}{15}\right)\right)$, which has many terms that cancel, starting with $-\frac{1}{7}$ and $\frac{1}{7}$ and ending with $-\frac{1}{13}$ and $\frac{1}{13}$. Eliminating these leaves $\frac{1}{2}\left(\frac{1}{5} + \frac{1}{6} - \frac{1}{14} - \frac{1}{15}\right) = \frac{1}{2} \cdot \frac{42+35-15-14}{210} = \frac{48}{420} = \frac{12}{105} = \frac{4}{35}$.

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19. If $y = -4x^3 + x^2 - x - 3$, evaluate $\frac{dy}{dx}$ at the point$(-2, 35)$.

$y' = -12x^2 + 2x - 1$, so $y'(-2) = -12 \cdot 4 - 4 - 1 = -53$.

20. What is the solution, in the form (c, d, f), of the system of equations $-c + d + f = -15$, $c - d + f = 1$, and $c + d - f = 15$?

Adding the equations in pairs gives $2f = -14$, $2d = 0$, and $2c = 16$, for an answer of $(8,0,-7)$.

21. When 7 liters of 7% sugar solution are mixed with 49 liters of 72% sugar solution, what percent of the final solution is sugar? Express your answer as a decimal, e.g. 7.89%.

Because the solutions have volumes in the ratio 1:7, the final mixture's percentage will be $\frac{1}{8}$ of the way from 72 to 7, which is a distance of $72 - 7 = 65$, for an answer of $72 - \frac{65}{8} = 72 - 8\frac{1}{8} = 64 - \frac{1}{8} = 63\frac{7}{8} = 63.875$.

22. How many months are in $49\frac{1}{4}$ years?

$49\frac{1}{4} \times 12 = 49 \cdot 12 + 3 = 600 - 12 + 3 = 591$

23. A right triangle with an angle measuring $45°$ has an area of 3 square meters. What is the perimeter, in meters, of the triangle?

It's a 45-45-90 triangle, so the area is $A = \frac{1}{2}s^2 = 3$, so that $s = \sqrt{6}$ and the hypotenuse is $\sqrt{2}\sqrt{6} = \sqrt{12} = 2\sqrt{3}$, for an answer of $2\sqrt{6} + 2\sqrt{3}$.

24. The average score on last week's Geometry quiz was 35.4. Excluding my score, the average of the other 21 students was 35.1. What was my score?

The total points scored were $22 \cdot 35.4 = 778.8$, and the total scored by everyone else was $21 \cdot 35.1 = 737.1$, for an answer of $778.8 - 737.1 = 41.7$.

25. Which quadrant does the line $4x - y = -4$ not pass through?

The intercepts are $(-1,0)$ and $(0,4)$, so the answer is IV.

26. A right triangle has legs measuring 14 and 36. What is the cosecant of the smallest angle in the triangle?

The cosecant is the reciprocal of the sine, and the smallest angle will be opposite the smallest side, the 14, so the answer will be $\frac{hypotenuse}{14} = \frac{\sqrt{14^2+36^2}}{14} = \frac{2\sqrt{7^2+18^2}}{14} = \frac{\sqrt{49+324}}{7} = \frac{\sqrt{373}}{7}$.

27. How many positive integers are factors of 7764?

$7764 = 2^2 \cdot 3^1 \cdot 647^1$, so it has $(2+1)(1+1)(1+1) = 3 \cdot 2 \cdot 2 = 12$ factors.

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28. A cube of purple plastic is painted orange on four sides and then cut into 1728 congruent cubes. What is the largest possible number of these smaller cubes that could be orange on exactly one side?

1728 means that the cube has been sliced into twelfths in each dimension. The four faces can either include two triplets that each share a common vertex or they can be four that make a band around the cube. Either way, the centers of the four faces each contain $10^2 = 100$ small cubes that are orange on one face. The two triplets collectively have six edges between orange and purple, while the band has eight, so the band will produce more cubes that are orange on one face: $8 \times 10 = 80$. Thus our answer is $4 \cdot 100 + 80 = 480$.

29. The backpacker has completed her dayhike to the summit near her camp, and now must filter water at the stream before returning to her camp. If the camp is at the point $(8, -1)$, the summit is at the point $(-7, -6)$, and the stream follows the line $y = 8x - 7$, what is the shortest distance she can hike?

Because the river is between the camp and the summit, we'll just go straight towards camp, getting water along the way, for an answer of $\sqrt{\left(8-(-7)\right)^2 + \left(-1-(-6)\right)^2} = \sqrt{15^2 + 5^2} = 5\sqrt{3^2 + 1^2} = 5\sqrt{10}$.

30. At 9PM on Halloween, as I'm about to turn out our porch light, the last four Trick-or-Treaters arrive at my house. I have nine Kit-Kats left, and I decide to randomly distribute them among the four kids without regard for fairness, except to make sure each kid gets at least one Kit-Kat. In how many ways can I distribute the candy?

Start by giving each kid one Kit-Kat, leaving $9 - 4 = 5$ to distribute randomly. They could go 5-0-0-0 four ways, 4-1-0-0 twelve ways, 3-2-0-0 twelve ways, 3-1-1-0 twelve ways, 2-2-1-0 twelve ways, or 2-1-1-1 four ways, for a total of 56 ways.