2016 Ciphering Time Trials
Solutions

1. **What is the volume, in cubic meters, of a right rectangular pyramid with base edges measuring 8 m and 12 m and a height of 5 m?**

   $V = \frac{1}{3}Bh = \frac{1}{3}lwh = \frac{1}{3} \cdot 8 \cdot 12 \cdot 5 = 8 \cdot 4 \cdot 5 = 8 \cdot 20 = 160$

2. **Express the hexadecimal numeral $8AD4_{16}$ as a binary numeral.**

   Each digit (0-F is like 0-15) in base 16 is equivalent to four digits in base 2, so the equivalent binary numeral is $1000 = 8$, $1010 = 10 = A$, $1101 = 13 = D$, and $0100 = 4$, for an answer of 1000101011010100.

3. **What is the sum of the squares of the roots of $2r^2 - r + 3 = 0$?**

   Let the roots be s & t. We can then know that $s + t = -\frac{a}{b} = -\frac{-1}{2} = \frac{1}{2}$ and $st = \frac{c}{a} = \frac{3}{2}$. What we want to know is $s^2 + t^2$, which seems a bit like $(s + t)^2 = s^2 + 2st + t^2$, so we really want $s^2 + t^2 = (s + t)^2 - 2st = \left(\frac{1}{2}\right)^2 - 2\left(\frac{3}{2}\right) = \frac{1}{4} - 3 = -\frac{11}{4}$.

4. **What is the equation of the axis of symmetry of the parabola with equation $x = 2y^2 - 20y + 38$?**

   This parabola is horizontal, not vertical, so its axis of symmetry will be $y = -\frac{b}{2a} = -\frac{-20}{2 \cdot 2} = \frac{20}{4} = 5$.

5. **What is the sum of the positive integers less than 17?**

   We're adding up the first 16 numbers, so the sum will be $\frac{16 \cdot 17}{2} = 8 \cdot 17 = 80 + 56 = 136$.

6. **In set of five integer test scores from 0 to 100 inclusive, the mean, median, and mode are all 80. What is the largest possible value of the range?**

   One possibility is to keep everything centered, getting the set 60, 80, 80, 80, 100, for a possible answer of $100 - 60 = 40$. However, the rightmost 80 could be increased, and if it were the 60 could be decreased, which would increase our range. It could go as high as 99, letting the 60 go as low as 41, for an answer of $100 - 41 = 59$.

7. **What is the area, in square meters, of a triangle with sides measuring 3 m, 5 m, and 6 m?**

   Using Heron's Formula gives $A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{7 \cdot 4 \cdot 2 \cdot 1} = 2\sqrt{14}$.

8. **On average, ten woodworkers can produce fifteen headboards in twelve hours. How many hours would it take for six woodworkers to produce six headboards?**

   The current time is 12 hours. We're multiplying the number of woodworkers by $\frac{6}{10} = \frac{3}{5}$, so we'll need to divide the time by this amount (more workers take less time & vice versa). Similarly, we're multiplying the amount of work by $\frac{6}{15} = \frac{2}{5}$, so we'll need to multiply the time by this amount as well (more work takes more time), for an answer of $12 \cdot \frac{5}{3} \cdot \frac{2}{5} = 4 \cdot 2 = 8$.

2016 Ciphering Time Trials
Solutions

9. A dragon senses that his hoard of 2500 coins contains exactly one counterfeit coin which is slightly lighter than the others. Fortunately, he has a very precise balance on which he can compare any two sets of coins. Using an optimal strategy, what is the greatest number of such comparisons that the dragon must be prepared to make in order to find the offending coin?

Any one weighing can determine if the light coin is in the left basket, the right basket, or neither (i.e. still in the unweighed coins). The optimal strategy is to make these three groups of coins the same size, so that we can discard two thirds of the coins each time. The first such weighing must narrow it down to 834 coins. The second gets us to 278 coins, then 93, 31, 11, 4, 2, and 1, for an answer of 8 weighings.

10. When the secret number is tripled and this result is then reduced by 758, the final result is 1345. What is the secret number?

Working backwards, three times the secret number must be $1345 + 758 = 2103$, so that the secret number is $\frac{2103}{3} = 701$.

11. A triangle has sides measuring 12 m, 14 m, and 6 m. The angle bisector of the largest angle is drawn, dividing the far side into two line segments. What is the length, in meters, of the larger of these two segments?

The two segments will be proportional to their adjacent sides, so our segment will be $\frac{12}{12+6} = \frac{12}{18} = \frac{2}{3}$ of 14, for an answer of $\frac{28}{3}$.

12. What is the sum of the positive even numbers less than 524?

We're looking for even numbers up to 522, which is $\frac{522}{2} = 261$ even numbers, for a sum of $261 \cdot 262 = 522 + 15660 + 52200 = 68382$.

13. What is the remainder when 1689 is divided by 24?

The standard division algorithm gives 70r9, for an answer of 9.

14. List the numbers below that are divisible by 3 but not 6.
14, 52, 789, 78, 66, 579, 41, 25, 678, 256, 39, 0

They need to be divisible by 3 but not 2, which means having digits that sum to 3, but not ending in an even digit. 14 is even, as is 52. 789 is not, and the digits add up to 24, so it is a multiple of 3. 78 is even, as is 66. 579 is not, and the digits add up to 21, so it is a multiple of 3. 41 is not even, but is also not a multiple of 3. The same is true of 25. 678 is even, as is 256. 39 is not, and it is a multiple of 3. 0 is even. Thus, our answer is 789, 579, and 39.

15. Cherie surrounds a rectangular picture of her daughter with a pretty rectangular border that is the same width on all sides of the picture. If the picture measures 20 cm by 25 cm, and the area of the border (NOT including the picture) is 826 cm^2, what is the perimeter, in centimeters, of the exterior edge of the border?

The border can be broken into rectangles adjacent to the picture and square corners connecting those rectangles. Thinking of it this way, the area of the border will be $90w + 4w^2 = 826$. Hoping for a "nice" answer, we tested 8 (too big) and then 7 (just right), rather than solving the quadratic. This makes the perimeter of the border $90 + 8w = 90 + 56 = 146$.

2016 Ciphering Time Trials
Solutions

16. Express 60135.79 in scientific notation rounded to three significant digits.

The three significant digits are 6, 0, and 1 (with the 3 causing the 1 to not round up), so the number is 60100. In scientific notation, this is 6.01×10^4.

17. What is the area, in square meters, of a right triangle with legs measuring 42 m and 68 m?

$A = \frac{1}{2}bh = \frac{1}{2} \cdot 42 \cdot 68 = 21 \cdot 68 = 68 + 1360 = 1428$

18. Two computers each choose a random number between 0 and 1. What is the probability that the smaller number is more than one-third of the larger number?

Each number can range from 0 to 1, so the sample space can be considered to be a square in the Cartesian Plane. The region of this square above the line $y = \frac{1}{3}x$ and below the line $y = 3x$ is the "good" area. The "bad" area is two right triangles, and has a total area of $\frac{1}{3}$, so that our answer is $\frac{1-\frac{1}{3}}{1} = \frac{2}{3}$.

19. What is the slope of a line perpendicular to the line $2x - 5y = 7$ and passing through the point $(-53, -45)$?

The point is a red herring; this line has a slope of $m = -\frac{A}{B} = -\frac{2}{-5} = \frac{2}{5}$, so the slope of the perpendicular line will be the negative reciprocal, $-\frac{5}{2}$.

20. If $v(u) = 6u + 2$ and $t(s) = 3s - 9$, evaluate $v\left(t^{-1}\left(v^{-1}(t(10))\right)\right)$.

$t(10) = 3 \cdot 10 - 9 = 30 - 9 = 21$. $v^{-1}(21)$ is the value of u that would produce $v = 21$, so we need to solve $21 = 6u + 2$, getting $6u = 19$ and then $u = \frac{19}{6}$. $t^{-1}\left(\frac{19}{6}\right)$ is the value of s that would produce $t = \frac{19}{6}$, so we need to solve $\frac{19}{6} = 3s - 9$, getting $3s = \frac{19}{6} + \frac{54}{6} = \frac{73}{6}$ and then $s = \frac{73}{18}$. $v\left(\frac{73}{18}\right) = 6 \cdot \frac{73}{18} + 2 = \frac{73}{3} + \frac{6}{3} = \frac{79}{3}$.

21. If u is an angle in the fourth quadrant with a secant of -3, find the possible values of $\csc\frac{u}{2}$.

If the secant is -3, then the cosine is $-\frac{1}{3}$. We want the cosecant, so we'll need to find the sine. Because we know $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$, we can write $\sin x = \pm\sqrt{\frac{1-\cos(2x)}{2}}$, which also means $\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos(u)}{2}} = \pm\sqrt{\frac{1-\left(-\frac{1}{3}\right)}{2}} = \pm\sqrt{\frac{\frac{4}{3}}{2}} = \pm\sqrt{\frac{2}{3}}$. The cosecant is thus $\pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$.

22. What is the circumference, in meters, of a circle with an area of 725π m^2?

$A = 725\pi = \pi r^2$, so $r^2 = 725 = 25 \cdot 29$, giving $r = 5\sqrt{29}$. $C = 2\pi r = 2\pi \cdot 5\sqrt{29} = 10\pi\sqrt{29}$.

2016 Ciphering Time Trials
Solutions

23. A recursive sequence has first term $q_1 = 144$ and subsequent terms defined by $q_n = \frac{q_{n-1}}{n} + n$. What is the fourth term of this sequence?

$q_2 = \frac{q_1}{2} + 2 = \frac{144}{2} + 2 = 72 + 2 = 74$, $q_3 = \frac{q_2}{3} + 3 = \frac{74}{3} + \frac{9}{3} = \frac{83}{3}$, and $q_4 = \frac{q_3}{4} + 4 = \frac{\frac{83}{3}}{4} + 4 = \frac{83}{12} + \frac{48}{12} = \frac{131}{12}$.

24. When the digits of a positive two-digit integer are reversed to form a new positive two-digit integer, the result is exactly ten more than half of the original number. What is the smallest possible value of the original number?

If the original number is TU, which is equal to $10T + U$, the reversed number is $10U + T = \frac{1}{2}(10T + U) + 10$. This becomes $\frac{19}{2}U = 4T + 10$ or $19U = 8T + 20$. Clearly U must be even. Trying 2 doesn't work out, but $U = 4$ gives $T = 7$, for an answer of 74.

25. The probability that it rains tomorrow is $\frac{1}{3}$. If it rains, the probability that I go hiking is $\frac{1}{4}$, otherwise it's $\frac{1}{2}$. What is the probability I hike in the rain tomorrow?

To hike in the rain, it must rain $\left(\frac{1}{3}\right)$ and I must hike $\left(\frac{1}{4}\right)$, for a probability of $\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$.

26. A square is inscribed in a circle, which is inscribed in an equilateral triangle. If the triangle's area is $9\sqrt{3}$ m^2, what is the square's perimeter, in meters?

The triangle's area is $A = 9\sqrt{3} = \frac{s^2\sqrt{3}}{4}$, so that $s^2 = 36$ and thus $s = 6$. Drawing a 30-60-90 triangle with a vertex in the center of the circle makes its radius $r = \frac{3}{\sqrt{3}} = \sqrt{3}$, so the square's diagonal is $2\sqrt{3}$, making the square's sides $\frac{2\sqrt{3}}{\sqrt{2}} = \sqrt{6}$, for an answer of $4\sqrt{6}$.

27. What value(s) of w satisfy $\frac{2w+3}{4-w} = \frac{4w+5}{w+1}$?

Cross-multiplying gives $2w^2 + 5w + 3 = -4w^2 + 11w + 20$, which becomes $6w^2 - 6w - 17 = 0$. The Quadratic Formula gives $w = \frac{6 \pm \sqrt{6^2 - 4 \cdot 6(-17)}}{2 \cdot 6} = \frac{6 \pm \sqrt{36 + 24 \cdot 17}}{12} = \frac{6 \pm 2\sqrt{9 + 6 \cdot 17}}{12} = \frac{3 \pm \sqrt{111}}{6}$.

28. What is the surface area of a right circular cylinder with a base radius of 4 m and a height of 15 m?

The surface area is two circles with a rectangle wrapped around their circumferences, so its value is $2\pi r^2 + 2\pi rh = 2\pi \cdot 4^2 + 2\pi \cdot 4 \cdot 15 = 32\pi + 120\pi = 152\pi$.

2016 Ciphering Time Trials
Solutions

29. What is the maximum value of the function $p(m) = 3m^3 - 5m^2 + m$ on the interval $m \in [-2, 2]$?

The maximum could be where the derivative is 0, or at either end of the interval. The ends of the interval are at $(-2, -46)$ and $(2,6)$. The derivative is $9m^2 - 10m + 1 = (9m - 1)(m - 1)$, so it has zeroes at $m = 1$ and $m = \frac{1}{9}$. Knowing the rough shape of the conic, the $m = 1$ will be a local minimum that we'll ignore, and the $m = \frac{1}{9}$ will be a local maximum at ($\frac{1}{9}$, some small positive value), making our answer 6.

30. What is the largest integer value of w that satisfies $3 \cdot 2^{6w} - 926 \leq 10^{10}$?

We can rewrite this as $3 \cdot 2^{6w} \leq 10^{10} + 926$, then $2^{6w} \leq \frac{10^{10}+926}{3} \approx 3.3 \cdot 10^9$. Knowing that $2^{10} = 1024 \approx 10^3$, we can determine that $s^{30} \approx 10^9$. This would correspond to $w = 5$, and it's definitely less than 3.3×10^9. Increasing to $w = 6$ would multiply by $2^6 = 64$, which would definitely be too high, making our answer 5.