2015 Ciphering Time Trials
Solutions

1. **Evaluate: $\frac{7!9!}{5!2!8!}$**

   $\frac{7!9!}{5!2!8!} = \frac{7\cdot 6\cdot 9}{2} = 7\cdot 3\cdot 9 = 21\cdot 9 = 189$

2. **A regular tetrahedron has a surface area of 400 m^2. What is the surface area, in square meters, of a larger regular tetrahedron each edge of which is twice as long as those of the original tetrahedron?**

   If every edge of the new tetrahedron is twice as long as the old one, then each face has $2^2 = 4$ times the area of each of the old ones, so that the new total surface area will be $4\cdot 400 = 1600$.

3. **When four identically unfair coins are flipped, the probability of getting exactly two heads is the same as the probability of getting exactly three heads. What is the probability that exactly one of the four coins shows heads?**

   If the probability of heads is h, then tails must be $1-h$, and the probability of getting two heads must be $4c2\cdot h^2(1-h)^2 = 4c3\cdot h^3(1-h)$, which is the probability of getting three heads. This relation becomes $3(1-h) = 2h$, then $3 - 3h = 2h$ and $3 = 5h$, giving $h = \frac{3}{5}$.

   Thus, the probability of getting one head out of four is $4c1\cdot h(1-h)^3 = 4\cdot\frac{3}{5}\left(\frac{2}{5}\right)^3 = \frac{4\cdot 3\cdot 8}{5^4} = \frac{96}{625}$.

4. **What is the circumference, in meters, of a circle with an area of 150π m^2?**

   $A = 150\pi = \pi r^2$, so $150 = r^2$ and $r = \sqrt{150} = 5\sqrt{6}$. $C = 2\pi r = 2\pi\cdot 5\sqrt{6} = 10\pi\sqrt{6}$.

5. **What are the coordinates, in the form (x, y), of the point of intersection of the lines $y = 2x - 3$ and $y = -3x + 22$?**

   Subtracting them gives $0 = 5x - 25$, which becomes $25 = 5x$, giving $x = 5$ and thus $y = 2\cdot 5 - 3 = 10 - 3 = 7$, for an answer of $(5,7)$.

6. **Anne's dad is making goodie bags for her birthday party. He has six identical Snix candy bars to distribute among four identical bags, but gives no thought to fairness. How many different sets of four bags could he create?**

   He could arrange them 6-0-0-0, 5-1-0-0, 4-2-0-0, 3-3-0-0, 4-1-1-0, 3-2-1-0, 2-2-2-0, 3-1-1-1, or 2-2-1-1, for an answer of 9.

7. **The product of two positive numbers is 720, and the ratio between them is 4:5. What is the sum of the two numbers?**

   Let the two numbers be $4b$ and $5b$, so that their product is $4b\cdot 5b = 20b^2 = 720$, which becomes $b^2 = 36$, so that $b = \pm 6$. Because they are positive numbers, $b = 6$, $4b = 24$, and $5b = 30$, for an answer of $24 + 30 = 54$.

8. **What are the coordinates, in the form (x, y), of the center of the graph of $2x^2 - 3y^2 + 4x + 9y = 100$?**

   Completing the square gives $2(x+1)^2 - 3\left(y - \frac{3}{2}\right)^2 = 100 + 2 - \frac{27}{4}$, so the center will be $\left(-1, \frac{3}{2}\right)$.

9. **List which of the following numbers is/are divisible by 12.**
486, 7216, 15204, 901355, 1348, 906, 5353, 4512, 805, 723, 41

To be divisible by 12, you must be divisible by 3 and by 4. To be divisible by 4, your last two digits must be divisible by 4 (because any multiple of 100 will be divisible by 4). Thus, we need only consider 7216, 15204, 1348, and 4512. To be divisible by 3, the sum of the digits must be a multiple of 3, so we're down to 15204 and 4512.

10. **Express in simplest radical form: $\sqrt[3]{540}$**

$\sqrt[3]{27 \cdot 20} = 3\sqrt[3]{20}$

11. **What is the seventh term of a geometric sequence with first term 5 and common ratio 3?**

The seventh term will be six ratios away from the first term, so it will be $5 \times 3^6 = 5 \times 27^2 = 5 \times 729 = 3645$.

12. **You have 47 fence sections that are each three meters long, and a huge supply of connectors, some of which allow the fence to continue straight, some of which allow the fence to make a right-angled turn. What is the largest area you can completely enclose using these supplies?**

Essentially, we need to make some kind of right-angled figure from the 47 sections. Thinking about it, if there is a section on the right side, there must be a corresponding section on the left side, and the same thing is true for the upper and lower sides. So, we can only use an even number of fence sections, which means we can only use 46 sections. We need to use an even number on the sides and an even number on the top & bottom. Generally speaking, if you're building a rectangular enclosure with a fixed perimeter (46 sections), you'd like to build a square. With 46 sections, we can't make the four sides equal, but we could do 11, 12, 11, and 12, which would have an area of $11 \cdot 12 \cdot 3^2 = 132 \cdot 9 = 1188$.

13. **What is the sum of the number of faces on a dodecahedron, the number of inches in a foot, and the number of months in a year?**

A dodecahedron has 12 faces which are pentagons, there are 12 inches in a foot, and there are 12 months in a year, so the sum is $12 + 12 + 12 = 36$.

14. **What is the discriminant of the equation $0 = 2x^2 - 3x - 4$?**

The discriminant is $b^2 - 4ac = (-3)^2 - 4 \cdot 2(-4) = 9 + 32 = 41$.

15. **A car drives around an elliptical track with equation $x^2 + 2y^2 = 243$. If the car is at position $x = -15$ in the third quadrant and the car's horizontal velocity is $\frac{dx}{dt} = 10$, what is the car's vertical velocity $\frac{dy}{dt}$?**

We can write $(-15)^2 + 2y^2 = 243$, then $225 + 2y^2 = 243$, which becomes $2y^2 = 243 - 225 = 18$, then $y^2 = 9$, giving $y = \pm 3$. Because we're in the third quadrant, $y = -3$. Taking the derivative of the original relationship with respect to time gives $2xx' + 4yy' = 0$, which becomes $2yy' = -xx'$, then $y' = -\frac{xx'}{2y}$. Substituting our known values gives $y' = -\frac{(-15)(10)}{2 \cdot (-3)} = -5 \cdot 5 = -25$.

2015 Ciphering Time Trials
Solutions

16. When the secret number is decreased by 38 and this result is then tripled, the final result is 369. What is the secret number?

Working backwards, the pre-triple number must have been $369 \div 3 = 123$, so that the secret number is $123 + 38 = 161$.

17. What is the area, in square meters, of a 30-60-90 triangle with a long leg measuring 9 m?

If the long leg is 9, the short leg is $\frac{9}{\sqrt{3}} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}$, and the area is $\frac{1}{2} \times 9 \times 3\sqrt{3} = \frac{27\sqrt{3}}{2}$.

18. What is the smallest positive four-digit integer that is congruent to 7 in mod 9 and 3 in mod 14?

We're looking for numbers that are two less (seven more) than a multiple of 9 and three more (11 less) than a multiple of 14. Checking the 14's is quicker: 3, 17, 31, 45, 59, 73, 87, 101, 115 – aha! So 115 fits the criteria, but isn't big enough. We need a number that leaves a remainder of 115 when divided by the LCM of 9 and 14, which is $9 \times 14 = 126$. 126 is close to 125, and $8 \times 125 = 1000$, so $8 \times 126 = 1008$, and $1008 + 115 = 1123$, which should be the answer.

19. Mahen can write a math contest in ten hours, and Gregg could write one in fifteen hours. If they work together, how many minutes would it take the two of them to write a math contest?

Their combined speed is $\frac{1}{10} + \frac{1}{15} = \frac{3}{30} + \frac{2}{30} = \frac{5}{30} = \frac{1}{6}$ of a contest per hour, so it will take them 6 hours to write a contest together, which is $6 \times 60 = 360$ minutes.

20. When two cards are drawn from a standard 52-card deck, what is the probability that neither one is a face card (a Jack, Queen, or King)?

The "first" card could be any of the other ten ranks in any of the four suits, for a probability of $40/52$. The "second" card can be any of the 39 remaining non-face cards, out of all of the 51 remaining cards, for a probability of $\frac{39}{51}$, for a total probability of $\frac{40}{52} \times \frac{39}{51} = \frac{10}{13} \times \frac{13}{17} = \frac{10}{17}$.

21. A planar diagram is composed of three circles and two lines. What is the largest number of regions into which the plane could be divided?

Drawing this as large as you can on paper counting regions as you create them is probably the best approach to this problem. The first circle divides the initial region into two. The second circle can divide both of these regions into two, for a total of four regions. The third circle can divide all of these regions into two, for a total of eight regions. The first line can pass through six of these regions, dividing each of them into two, for six additional regions and a total of $8 + 6 = 14$ regions. At this point, it's easy to think you created seven new regions, but the region before you enter the circles and the region after you enter the circles are the same region. That will not be the case with the second line, which can divide eight regions, thus creating eight additional regions for a total of $14 + 8 = 22$ regions.

22. Express the base five number 4124_5 as a base ten number.

From the right, the digits in base 5 represent $5^0 = 1$, $5^1 = 5$, $5^2 = 25$, etc., so this number is $4 \times 125 + 1 \times 25 + 2 \times 5 + 4 \times 1 = 500 + 25 + 10 + 4 = 539$.

23. How many subsets of the set of two-digit perfect squares contain exactly two even numbers?

The two-digit perfect squares are 16, 25, 36, 49, 64, and 81. Three of these are even, so there are $3c2 = 3$ ways to pick two of them. Of the three odd numbers, each may choose whether or not to join a set, for an answer of $3 \times 2^3 = 3 \times 8 = 24$.

24. Simplify $(3x^4 - 6x^3 + 7x^2 + 10x - 48) \div (x - 2)$.

$3x^4 \div x = 3x^3$, and $3x^3(x-2) = 3x^4 - 6x^3$, leaving $7x^2 + 10x - 48$. $7x^2 \div x = 7x$, and if there is no remainder $-48 \div (-2) = 24$, for an answer of $3x^3 + 7x + 24$. Double-checking, $(x-2)(7x+24) = 7x^2 + 24x - 14x - 48 = 7x^2 + 10x - 48$, so our answer is right.

25. What are the coordinates, in the form (x, y), of the leftmost x-intercept of the graph of $y = 2x^2 - 21x + 54$?

Factoring gives $0 = (2x - 9)(x - 6)$ with roots of $\frac{9}{2}$ and 6, for an answer of $(\frac{9}{2}, 0)$.

26. In the cryptarithm $AB + AC = CA$, each letter represents a different digit (0-9), so if one A is a 2, all A's are 2's and B cannot be 2. What is the largest possible value of the three-digit number ABC?

For a large answer, we'd like a large value of A, which might be as high as 4 if C is 8 or 9. Either way, $B + C = A$ in the units column, so there will be a carry into the next column, so C would have to be 9, in which case B would have to be 5, so that $45 + 49 = 94$ and our answer is 459.

27. What is the volume, in cubic meters, of a right rectangular pyramid with edges measuring 2 m, 7 m, and 1 m?

7 must be the "slant" edges, with 1 and 2 being the base edges. Thus, the height of the pyramid is $\sqrt{7^2 - \left(\frac{\sqrt{5}}{2}\right)^2} = \sqrt{49 - \frac{5}{4}} = \sqrt{\frac{191}{4}} = \frac{\sqrt{191}}{2}$, and the volume is $\frac{1}{3} \times \frac{\sqrt{191}}{2} \times 2 \times 1 = \frac{\sqrt{191}}{3}$.

28. Evaluate as a mixed number: $6\frac{7}{8} \div 2\frac{3}{4}$

$6\frac{7}{8} \div 2\frac{3}{4} = \frac{55}{8} \div \frac{11}{4} = \frac{55}{8} \times \frac{4}{11} = \frac{5}{2} = 2\frac{1}{2}$

29. A 45-45-90 triangle has a hypotenuse measuring 8 m. What is the radius, in meters, of its inscribed circle?

The legs will be $\frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$. Drawing radii to each side and segments from the center of the circle to each vertex creates two sets of congruent triangles that allows us to break the legs into r and $4\sqrt{2} - r$, so that the hypotenuse would be $8 = \left(4\sqrt{2} - r\right) + \left(4\sqrt{2} - r\right) = 8\sqrt{2} - 2r$, so that $2r = 8\sqrt{2} - 8$ and $r = 4\sqrt{2} - 4$.

30. What is the range of the mean, median, and mode of the data set $\{\mathbf{1, 56, 7, 81, 35, 1, 68, 23, 61}\}$?

In ascending order, the set is 1, 1, 7, 23, 35, 56, 61, 68, 81, so the mode is 1 and the median is 35. The mean is $\frac{1+1+7+23+35+56+61+68+81}{9} = \frac{32+91+129+81}{9} = \frac{123+210}{9} = \frac{333}{9} = 37$, so the range is $37 - 1 = 36$.