2014 Ciphering Time Trials
Solutions

1. **A regular 20-gon has vertices labeled in clockwise order from A to T. A line drawn through vertex F and the center also passes through another vertex. What letter is the label of the vertex that the line also passes through?**

   Because it's a 20-gon, opposite vertices will be $20 \div 2 = 10$ apart, so that F is across from GHIJKLMNO…P.

2. **If Set $\boldsymbol{Q}$ is $\{2, 4, 6, 8, 10, 12, 14\}$ and Set $\boldsymbol{P}$ is $\{3, 6, 9, 12, 15, 18\}$, what is $\boldsymbol{P \cap \bar{Q}}$?**

   $P \cap \bar{Q}$ is the intersection of things that are in P and things that are NOT in Q. Going through P, 3 works, 6 doesn't, 9 works, 12 doesn't, 15 works, and 18 works, for an answer of {3, 9, 15, 18}.

3. **What is the sum of the mode and median of the data set $\{2, 89, 73, 4, 8, 79, 0, 2, 34, 6\}$?**

   Writing the set in ascending order, {0, 2, 2, 4, 6, 8, 34, 73, 79, 89}, makes both clear. The mode is 2 and the median is $\frac{6+8}{2} = \frac{14}{2} = 7$, for an answer of $2 + 7 = 9$.

4. **Express the base 7 number 1234_7 as a base 10 number.**

   In base 7, each digit is a power of 7, just like in base 10, each digit is a power of 10. Thus, our answer is $4 \times 7^0 + 3 \times 7^1 + 2 \times 7^2 + 1 \times 7^3 = 4 \times 1 + 3 \times 7 + 2 \times 49 + 1 \times 343 = 4 + 21 + 98 + 343 = 466$.

5. **K kilograms of ketchup cost exactly N nickels. How many kilograms of ketchup can I buy with D dollars?**

   Thinking of it in cents, we were able to buy K kilograms for $5N$ cents, but now we have $100D$ cents, so we should multiply K by $\frac{100D}{5N} = \frac{20D}{N}$, for an answer of $\frac{20DK}{N}$.

6. **A hexagon has sides measuring 9 m, 16 m, 25 m, 36 m, 49 m, and 64 m, in no particular order. One of its diagonals has an integer length when measured in meters. What is the largest possible measure of that diagonal, in meters?**

   Using reasoning related to the Triangle Inequality, the total perimeter of the hexagon is $9 + 16 + 25 + 36 + 49 + 64 = 50 + 49 + 100 = 199$. A diagonal will divide this perimeter into two parts, each of which much satisfy the "Polygon Inequality"; the diagonal must be shorter than the sum of all of the sides on each of the two parts of the polygon. To get the longest possible diagonal, we'd like to have the two parts of the polygon have equal sums, meaning they'd each be half of the original. As it turns out, we can have one part be the 36 and 64 (100), and the other part be the 9, 16, 25, and 49 (99), so that our diagonal can have a length of 98.

7. **What is the perimeter, in meters, of an equilateral triangle with an area of $\frac{4\sqrt{3}}{9}$ m²?**

   The area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4} = \frac{4\sqrt{3}}{9}$, which becomes $s^2 = \frac{16}{9}$, giving $s = \frac{4}{3}$. The perimeter is $3s = 3 \times \frac{4}{3} = 4$.

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8. When two identically unfair coins are flipped, the non-zero probability of getting two heads is the same as the probability of getting one head. What is the probability of getting no heads?

If the probability of heads on a single coin is p, then we can write $p^2 = 2p(1-p) = 2p - 2p^2$, which becomes $3p^2 = 2p$, then $p = \frac{2}{3}$. Thus, the probability of getting no heads is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

9. If $\cos r = \frac{1}{5}$, what is the maximum possible value of $\cos 2r$?

$\cos 2r = 2\cos^2 r - 1 = 2 \times \frac{1}{25} - 1 = \frac{2}{25} - 1 = -\frac{23}{25}$

10. Evaluate: $\frac{1}{2} + \frac{3}{4} \div \frac{5}{6}$

$\frac{1}{2} + \frac{3}{4} \div \frac{5}{6} = \frac{1}{2} + \frac{3}{4} \times \frac{6}{5} = \frac{1}{2} + \frac{3}{2} \times \frac{3}{5} = \frac{1}{2} + \frac{9}{10} = \frac{5}{10} + \frac{9}{10} = \frac{14}{10} = \frac{7}{5}$

11. What is the 32nd term of an arithmetic sequence with first term 109 and common difference 87?

The 32nd term is 31 differences from the first term, so will be $109 + 87 \times 31 = 109 + 2697 = 2806$.

12. What is the solution, in the form (c, d), of the system of equations $4c - d = 26$ and $2c + 3d = 17$?

Subtracting the first equation from twice the second gives $7d = 8$, so that $d = \frac{8}{7}$. This means $4c - \frac{8}{7} = 26$, which becomes $4c = \frac{182}{7} + \frac{8}{7} = \frac{190}{7}$, giving $c = \frac{190}{4\times7} = \frac{95}{2\times7} = \frac{95}{14}$, for an answer of $\left(\frac{95}{14}, \frac{8}{7}\right)$.

13. What is the product of the smallest positive three-digit integer with three distinct digits and the largest two-digit integer with two distinct digits?

$102 \times 98 = (100+2)(100-2) = 100^2 - 2^2 = 10000 - 4 = 9996$

14. Express the solution to the system of equations $u + t = 7$, $u + r = 11$, and $t + r = 13$ as an ordered triple in the form (u, t, r).

Adding the three equations gives $2u + 2t + 2r = 31$, which becomes $u + t + r = \frac{31}{2}$. Subtracting each of the original equations from this gives $r = \frac{31}{2} - \frac{14}{2} = \frac{17}{2}$, $t = \frac{31}{2} - \frac{22}{2} = \frac{9}{2}$, and $u = \frac{31}{2} - \frac{26}{2} = \frac{5}{2}$, for an answer of $\left(\frac{5}{2}, \frac{9}{2}, \frac{17}{2}\right)$.

15. What is the sum of the perfect squares between 100 and 300, inclusive?

The sum of the first n perfect squares is $\frac{n(n+1)(2n+1)}{6}$. $100 = 10^2$ and $17^2 = 289$, so our answer should be $\frac{17\times18\times35}{6} - \frac{9\times10\times19}{6} = 17 \times 3 \times 35 - 3 \times 5 \times 19 = 15 \times (17 \times 7 - 19) = 15 \times (119 - 19) = 15 \times 100 = 1500$.

16. If $u(v) = 2^v$ and $w(v) = v^2$, evaluate $u(10) - w(10)$.

$u(10) - w(10) = 2^{10} - 10^2 = 1024 - 100 = 924$

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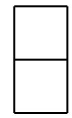
17. What is the area, in square meters, of a circle with a circumference of 11π m?

If the circumference is 11π, then the diameter is 11, so the radius is $\frac{11}{2}$, and the area is $\left(\frac{11}{2}\right)^2 \pi = \frac{121\pi}{4}$.

18. In the figure to the right composed of two unit squares, how many paths of length five are there from the upper left corner to the lower right corner? It is okay for such a path to use the same line segment multiple times.



The shortest paths (length three) consist of some permutation of RDD (right, down, and down). The length five paths still include RDD, but can either include DU or RL. However, the U or L cannot come before the first D or R, nor can it come after the last one. In the first case, the DDDU can only have two orders: DUDD or DDUD. For this case, the R could be in any of five positions, so there are $2 \times 5 = 10$ of these paths. For the second case, the RRL can only have one order: RLR. For this case, the two D's could be together (four cases) or separate ($4c2 = 6$ cases), for another $4 + 6 = 10$ cases and an answer of $10 + 10 = 20$.

19. In the system of equations $2s - 3r = 4$ and $Ks + 6r = 5$, there is a value of the constant K for which there is no solution to the system. What is that value of K?

Essentially, this requires the two lines to be parallel, so we need $\frac{2}{3} = -\frac{K}{6}$. Cross-multiplying gives $12 = -3K$, then $K = -4$.

20. A hexagon has a perimeter of 25 m and an area of 10 m^2. A larger, similar hexagon has an area of 45 m^2. What is the perimeter, in meters, of the larger hexagon?

If all of the dimensions of a figure are multiplied by r, the area of the figure will be multiplied by r^2. The ratio of our areas is $\frac{45}{10} = \frac{9}{2}$, so the ratio of the perimeters will be $\sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$ for an answer of $25 \times \frac{3\sqrt{2}}{2} = \frac{75\sqrt{2}}{2}$.

21. The point $(4, 3)$ is reflected across the line $y = 2$ to Point F, which is then rotated $270°$ clockwise around the point $(-1, 0)$ to Point G, which is then reflected across the line $y = x + 1$ to Point H. What are the coordinates, in the form (x, y), of Point H?

(4,3) is $3 - 2 = 1$ away from $y = 2$, so will reflect to be one away on the other side, at the point (4,1) (point F). (4,1) is $4 - (-1) = 4 + 1 = 5$ to the right and $1 - 0 = 1$ above the point $(-1{,}0)$, so after rotating 90° counter-clockwise it will be 5 above and 1 to the left of $(-1{,}0)$ at $(-2{,}5)$ (point G). $(-2{,}5)$ is $5 - (-1) = 6$ above (and/or to the left of) $y = x + 1$, so it will reflect to be 6 below (and/or to the right of) the line at $(4, -1)$.

22. What value(s) of b satisfy $3 + 5b = 8b - 11$?

$3 + 5b = 8b - 11$ becomes $14 = 3b$, giving $b = \frac{14}{3}$.

23. A triangle has two sides measuring 12 m and 16 m, with an angle of $60°$ between the two. What is the length, in meters, of the third side?

The law of cosines gives $c^2 = 12^2 + 16^2 - 2 \times 12 \times 16 \cos 60° = 144 + 256 - 12 \times 16 = 400 - 192 = 208$, so $c = \sqrt{208} = 2\sqrt{52} = 4\sqrt{13}$.

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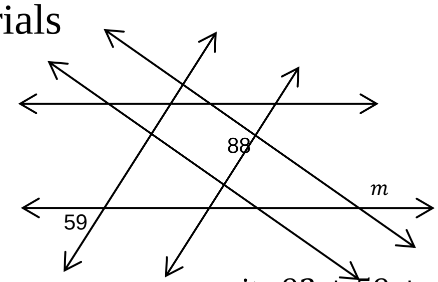

24. The figure to the right is composed of three pairs of parallel lines, and three angle measures are given in degrees. What is the value of m?

The triangles have angles of $180 - 88 = 92$, 59, and $180 - m$, so we can write $92 + 59 + (180 - m) = 180$, which becomes $m = 92 + 59 = 151$.

25. Two numbers sum to 89 and differ by 57. What is the larger of the two numbers?

The larger of the two numbers will be $\frac{89+57}{2} = \frac{146}{2} = 73$.

26. There is more than one pair of numbers that have a greatest common factor of 6 and a least common multiple of 600. What is the smallest possible sum of two such numbers?

Both numbers must have prime factorizations that have $6 = 2^1 \times 3^1$ in common, but no more 2's or 3's (or other numbers) in common. Somewhere in the prime factorizations of the two numbers there must be $600 = 2^3 \times 3^1 \times 5^2$, which is two extra 2's and two extra 5's. All of these extra 2's and 5's could be on one of the two numbers, resulting in 6 and 600, or they could be split between the two. If they're split, both 2's must go to one number and both 5's must go to the other, resulting in $6 \times 4 = 24$ and $6 \times 25 = 150$, for an answer of 174.

27. Simplify: $\sqrt{\mathbf{28} + \sqrt{\mathbf{434} + \sqrt{\mathbf{28} + \sqrt{\mathbf{434} + \cdots}}}}$

You can write $x = \sqrt{28 + \sqrt{434 + x}}$, square it a couple of times, and find roots of the resulting quadratic, but in this case it's quicker to just use logic. Our answer must be bigger than $\sqrt{28} \approx 5$, and furthermore must be bigger than $\sqrt{28 + \sqrt{434}} \approx \sqrt{28 + 20} = \sqrt{(48)} \approx 7$, so let's try assuming the answer is 7. Is $7 = \sqrt{28 + \sqrt{434 + 7}} = \sqrt{28 + \sqrt{441}} = \sqrt{28 + 21} = \sqrt{49} = 7$? Yes, so the answer is 7.

28. Evaluate: $(\mathbf{2} + \mathbf{3})^{\mathbf{4+5-6}}$

$(2 + 3)^{4+5-6} = 5^3 = 125$

29. When I reverse the digits of my age, I get my daughter's age. In *J* years, where *J* is a counting number, this will be true again. What is the smallest possible value of *J*?

There are many possible ages, including 53 and 35, which differ by 18. The key to this problem is that the difference in their ages will be a constant as they age. Because of the digit reversal, the difference in their ages is nine times the difference in their digits, in this example $18 = (5 - 3) \times 9 = 2 \times 9$. In J years, we need the difference in the digits to be the same as it is now, so J could be 11, 22, 33, etc. The smallest possible value of J is thus 11.

30. For marching along roads, King Peter tries to arrange his army in rows of seven, but there is one person left over. If he organizes them in rows of six there are two people left over. When he organizes them in rows of eight, however, no one is left over. What is the smallest number of people greater than 10,000 that could be in his army?

6 and 8 have an LCM of 24, so there should be a number from 1-24 that satisfies those two constraints. 8, 16, and 24 work for the 8 constraint, with 8 working for the 6 constraint, too. Thus, we're looking for numbers that leave a remainder of 8 when divided by 24 and a remainder of 1 when dividing by 7. The LCM of 7 and 24 is 168, so there should be a number from 1-168 that satisfies both constraints. 8, 32, 56, 80, 104, 128, and 152 all satisfy the 24 constraint, with 8 satisfying the 7 constraint as well. Thus, we're looking for the smallest number less than 10,000 that leaves a remainder of 8 when divided by 168. $10{,}000 \div 168 = 59r88$, so $10{,}000 - 88 + 8 = 9920$ satisfies all the conditions, and $9920 + 168 = 10{,}088$ will as well.