2013 Ciphering Time Trials
Solutions

1. Evaluate: $-2-(-3)-(-5)(-7)-11$

$-2-(-3)-(-5)(-7)-11=-2+3-35-11=-45$

2. How many positive integers are factors of 24300?

$24300=243\times 2^2\times 5^2=2^2\times 5^2\times 3^5$, so factors can have from zero to two 2's (three options), from zero to two 5's (three choices), and from zero to five 3's (six choices), for a total of $3\times 3\times 6=54$ factors.

3. Two concentric circles have radii of 9 m and 16 m. What is the length of a chord of the larger circle that is tangent to the smaller circle?

Drawing radii to an endpoint and the midpoint of the chord produces a right triangle with a hypotenuse of 16 and a leg of 9, so half of the chord has a length of $\sqrt{16^2-9^2}=\sqrt{256-81}=\sqrt{175}=5\sqrt{7}$, so that the entire chord has a length of $10\sqrt{7}$.

4. How many edges does a dodecahedron have?

A dodecahedron has twelve faces, each of which is a pentagon. Thus, there are $12\times 5=60$ edges, but each edge is actually part of two faces, so we need to divide by two, getting $60\div 2=30$.

5. What are the coordinates, in the form (x,y), of the midpoint of the points $(27,-64)$ and $(49,36)$?

$\frac{27+49}{2}=\frac{76}{2}=38$ and $\frac{-64+36}{2}=-\frac{28}{2}=-14$, for an answer of $(38,-14)$.

6. How many subsets of the set of one-digit composite numbers contain at least one even number?

The one-digit composite numbers are 4, 6, 8, and 9, three of which are even. The desired subsets must have at least one of the three even elements, so there are $2^3-1=8-1=7$ ways to choose among the evens. In addition, the subsets can have or not have the 9 (two ways), for a total of $7\times 2=14$ subsets.

7. What is the name of the point where the perpendicular bisectors of the three sides of a triangle meet?

Each perpendicular bisector is equidistant from the adjacent vertices, so the intersection of the three is equidistant from all vertices, which makes it the center of the circumscribed circle, the "circumcenter".

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8. What is the sum of the 35 smallest positive even integers?

The sum of the first *n* even integers is $n(n+1)$, which is just twice the formula for the first *n* integers. $35 \times 36 = 70 \times 18 = 1260$

9. If $m(n) = (2n+3)(4n^2 - 5n - 6) + 7$, evaluate $m(3)$.

$$m(3) = (2 \times 3 + 3)(4 \times 3^2 - 5 \times 3 - 6) + 7 = (6+3)(4 \times 9 - 15 - 6) + 7$$
$$= 9(36 - 21) + 7 = 9 \times 15 + 7 = 135 + 7 = 142$$

10. What are the coordinates, in the form (x, y), of the vertex of the parabola with equation $y = 3x^2 - 36x + 17$?

The vertex is on the axis of symmetry, $x = -\frac{b}{2a} = -\frac{-36}{2\times3} = \frac{36}{6} = 6$, which we substitute to get $y = 3 \times 6^2 - 36 \times 6 + 17 = 3 \times 36 - 216 + 17 = 108 - 216 + 17 = -108 + 17 = -91$ and an answer of $(6, -91)$.

11. What is the 2013th term of an arithmetic sequence with a first term of 9876 and a common difference of -6?

The 2013th term is 2012 differences from the first term, so it will be $9876 - 6 \times 2012 = 9876 - 12072 = -2196$.

12. In a right triangle with legs measuring 4 m and 10 m, what is the cosine of the smallest angle?

The hypotenuse is $\sqrt{10^2 + 4^2} = \sqrt{100 + 16} = \sqrt{116} = 2\sqrt{29}$. The smallest angle is opposite the smallest side, so that its cosine will be $\frac{10}{2\sqrt{29}} = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}$.

13. Express in simplest radical form: $5\sqrt{432}$

$5\sqrt{432} = 5 \times 2\sqrt{108} = 10 \times 2\sqrt{27} = 20 \times 3\sqrt{3} = 60\sqrt{3}$

14. What are the eigenvalue(s) of the matrix $\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$?

The eigenvalue(s) (λ) will satisfy $\begin{vmatrix} 1-\lambda & -1 \\ 3 & 5-\lambda \end{vmatrix} = 0$, which gives $(1-\lambda)(5-\lambda) - (-1)(3) = 5 - 6\lambda + \lambda^2 + 3 = \lambda^2 - 6\lambda + 8 = 0$, which factors to $(\lambda - 4)(\lambda - 2) = 0$ with roots of 4 and 2.

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15. Express the base ten number 974_{10} in base eight.

The digits in base 8 represent $8^0 = 1$, $8^1 = 8$, $8^2 = 64$, $8^3 = 512$, … There is one 512 in 974, leaving 462. There are seven 64's in 462, leaving 14. There is one 8 in 14, leaving 6, for an answer of 1716_8.

16. Arrange the letters in order of descending value (e.g. DCBA):

$\boldsymbol{A = 5!}$ $\boldsymbol{B = \sqrt{98765}}$ $\boldsymbol{C = 4^4}$ $\boldsymbol{D = \frac{1.23\times10^4}{8.76\times10^5}}$

$A = 5! = 5 \times 4 \times 3 \times 2 = 20 \times 6 = 120$, $B = \sqrt{98765} > 300$, $C = 4^4 = 16^2 = 256$, and $D = \frac{1.23\times10^4}{8.76\times10^5} \approx \frac{1}{4\times10^1} = .025$, for an answer of BCAD.

17. What is the length, in meters, of the altitude to the shortest side of a triangle with sides measuring 20 m, 21 m, and 39 m?

Heron' s Formula gives $A = \sqrt{40 \times 1 \times 19 \times 20} = 20\sqrt{2 \times 19} = 20\sqrt{38}$, but we also know that $A = \frac{1}{2}Bh$, so $20\sqrt{38} = \frac{1}{2} \times 20h$, which becomes $h = 2\sqrt{38}$.

18. Bag U contains two red marbles and nine green marbles, while Bag V contains six red marbles and one green marble. A bag is chosen at random, and then a random marble is drawn from that bag. If the resulting marble turns out to be red, what is the probability that Bag U was selected?

The marble must be a red from Bag U $\left(\frac{1}{2} \times \frac{2}{11} = \frac{1}{11}\right)$ or a red from Bag V $\left(\frac{1}{2} \times \frac{6}{7} = \frac{3}{7}\right)$. The probability that it is from Bag U is $\frac{\frac{1}{11}}{\frac{1}{11}+\frac{3}{7}} = \frac{\frac{1}{11}}{\frac{40}{77}} = \frac{1}{11} \times \frac{77}{40} = \frac{7}{40}$.

19. What is the solution, in the form (g, h, j), of the system of equations $g + h = 3$, $h + j = 7$, and $g + j = 12$?

Adding the three equations gives $2g + 2h + 2j = 22$, which becomes $g + h + j = 11$. Subtracting each equation from this gives $j = 11 - 3 = 8$, $g = 11 - 7 = 4$, and $h = 11 - 12 = -1$, for an answer of $(4, -1, 8)$.

20. I own the first two books of the Perry's Hotter series, all of the Roar of the Lings trilogy, and the last book of the Peal of Whines series. If I wish to arrange them next to one another on a shelf, keeping books in the same series together, in how many ways can I do that?

There are $3! = 6$ ways to arrange the three series, $2! = 2$ ways to arrange the books in the first series, $3! = 6$ ways to arrange the books of the trilogy, and just $1! = 1$ way to arrange the single book of the last series, for an answer of $6 \times 2 \times 6 \times 1 = 72$.

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21. Simplify in terms of $i = \sqrt{-1}$: $(7i^2 - 6i^3)\left(3 - \frac{2}{i}\right)$

$(7i^2 - 6i^3)\left(3 - \frac{2}{i}\right) = (-7 + 6i)(3 + 2i) = -21 - 12 + 18i - 14i = -33 + 4i$

22. If Eddie can paint three houses in four days and Farah can paint five houses in six days, how many days would it take them to paint ten houses in a new cul-de-sac?

Eddie's speed is $\frac{3}{4}$ houses per day, while Farah's is $\frac{5}{6}$, so together their speed is $\frac{3}{4} + \frac{5}{6} = \frac{9+10}{12} = \frac{19}{12}$ houses per day. To paint ten houses would thus take $\frac{10}{\frac{19}{12}} = \frac{120}{19}$ days.

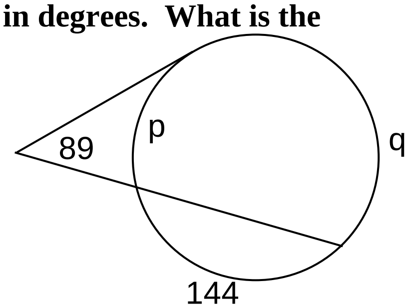
23. If $z(b) = \frac{b^2}{b-3}$, evaluate $z'(-4)$.

$z'(b) = \frac{(b-3)(2b) - b^2(1)}{(b-3)^2}$, so $z'(-4) = \frac{(-4-3)(2)(-4) - (-4)^2(1)}{(-4-3)^2} = \frac{-7(-8) - 16}{(-7)^2} = \frac{56-16}{49} = \frac{40}{49}$.

24. What is the sum of the positive two-digit integers that are not multiples of three?

There are 45 pairs of positive two-digit integers that each have a sum of $10 + 99 = 109$, for a total sum of $45 \times 109 = 4500 + 450 - 45 = 4905$. Now we need to subtract the multiples of 3. There are 15 pairs that each have a sum of $12 + 99 = 111$, for a total sum of $15 \times 111 = 1500 + 150 + 15 = 1665$. Subtracting these gives $4905 - 1665 = 3240$.

25. In the figure to the right, angle and arc measures are given in degrees. What is the value of q?



We can write $p + q = 360 - 144 = 216$ and $q - p = 89 \times 2 = 178$. Adding these gives $2q = 394$, so that $q = \frac{394}{2} = 197$.

26. If the domain and range of the function $s(t) = \sqrt{\frac{1-t}{t+2}}$ are both subsets of the real numbers, express the domain in interval notation.

The numerator switches sign when $t = 1$, while the denominator does so when $t = -2$. For large values of t, the numerator is negative while the denominator is positive, so that the quotient is negative and the square root is "broken". Similarly, for large negative values of t, the numerator is positive and the denominator is negative, also "breaking" the square root. In the middle, however, such as $t = 0$, the numerator and denominator are both positive, and these values are in the domain of the function. Note that $t = 1$ is in the domain, but $t = -2$ is not (we cannot divide by zero). Thus, the final answer is $(-2,1]$.

27. What is the population standard deviation of $\{3, 4, 6\}$?

We're looking for the square root of the mean of the squares of the deviations (from the average). The average is $\frac{3+4+6}{3} = \frac{13}{3}$, so the deviations are $-\frac{4}{3}$, $-\frac{1}{3}$, and $\frac{5}{3}$, with squares of $\frac{16}{9}$, $\frac{1}{9}$, and $\frac{25}{9}$. The average of these is $\frac{\frac{16}{9}+\frac{1}{9}+\frac{25}{9}}{3} = \frac{\frac{42}{9}}{3} = \frac{42}{27} = \frac{14}{9}$, with a square root of $\sqrt{\frac{14}{9}} = \frac{\sqrt{14}}{3}$.

28. Captain Even has eight identical earrings he wishes to distribute among his three lieutenants. If all distributions are equally likely, what is the probability that each lieutenant receives an even number of earrings (0 counts as even)?

The eight earrings can be distributed in $10c2 = \frac{10\times 9}{2} = 5 \times 9 = 45$ ways, while four pairs of earrings can be distributed in $6c2 = \frac{6\times 5}{2} = 3 \times 5 = 15$ ways, for a probability of $\frac{15}{45} = \frac{1}{3}$.

29. What value(s) of k satisfy $\frac{8k+7}{2k+3} = \frac{4k+5}{k-6}$?

Cross-multiplying gives $8k^2 - 41k - 42 = 8k^2 + 22k + 15$, which becomes $-63k = 57$, so that $k = -\frac{57}{63} = -\frac{19}{21}$.

30. A room is in the shape of a regular hexagonal prism with walls measuring 10 m long and a height of 3 m. If Randi has two square panels measuring 3 m on a side, what is the largest volume she can enclose in one of the room's corners?

Consider some shape that could be enclosed in the corner of the room. If that corner of the room were rotated $120°$ about itself twice, three of the enclosed shape would completely surround the corner. If we had the shape with the maximum area initially, this surrounding shape would have its maximum possible area, and vice versa. The largest possible surrounding shape is a hexagon with sides measuring 3, so the largest possible shape in the corner is one-third of this shape, which is two equilateral triangles with sides of 3, so that the answer is $2 \times \frac{3^2\sqrt{3}}{4} = \frac{9\sqrt{3}}{2}$.