2018 Team Scramble
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Easier Problems

1. What number is 5397 more than half of 9598?

$5397 + \frac{9598}{2} = 5397 + 4799 = 5397 + 4800 - 1 = 10{,}196$

2. Evaluate: $-50 - 9\big(-43 - (-76)\big) - 53$

$-50 - 9\big(-43 - (-76)\big) - 53 = -50 - 9 \cdot 33 - 53 = -103 - 297 = -400$

3. Evaluate: $59^2 - 41^2$

$59^2 - 41^2 = (59 - 41)(59 + 41) = 18 \cdot 100 = 1800$

4. Express $2.3\overline{4}$ as a fraction.

We can write $x = 2.3\overline{4}$, so $10x = 23.\overline{4} = 23\frac{4}{9} = \frac{211}{9}$, so $x = \frac{211}{9} \div 10 = \frac{211}{90}$.

5. What is the name for the place value where the digit 8 appears in the number 49023.71685?

The 8 is to the right of the decimal point, where the digits represent tenths, hundredths, thousandths, then ten-thousandths.

6. Evaluate: $\frac{36}{63} + \frac{7}{56}$

$\frac{36}{63} + \frac{7}{56} = \frac{4}{7} + \frac{1}{8} = \frac{32+7}{56} = \frac{39}{56}$

7. What value(s) of b satisfy $4b + 87 = 731$?

$4b + 87 = 731$ becomes $4b = 644$, so $b = 161$.

8. In which quadrant does the point $(-2357, -1928)$ lie?

x and y are negative, so the point is in the lower left quadrant, which is the third.

9. What is the equation, in slope-intercept form, of the line through the points $(2, -25)$ and $(-5, -4)$?

The slope of the line is $m = \frac{-4-(-25)}{-5-2} = \frac{21}{-7} = -3$, so the equation is $y = -3x + b$. Substituting gives $-25 = -3 \cdot 2 + b = -6 + b$, so $b = -19$, for an answer of $y = -3x - 19$.

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10. What is the midpoint of the line segment connecting the points $(8, -5)$ and $(-14, -9)$?

The midpoint will be $\left(\frac{8+(-14)}{2}, \frac{-5+(-9)}{2}\right) = \left(-\frac{6}{2}, -\frac{14}{2}\right) = (-3, -7)$.

11. When the digits of a positive two-digit integer are reversed, the result is a positive two-digit integer that is 54 less than the original number. What is the largest possible value of that original number?

When a two-digit number is reversed, the difference from the original is always 9 times the difference of the digits. In this case, we're looking for digits that differ by $\frac{54}{9} = 6$, which makes the largest possible original number 93.

12. What value(s) of w satisfy $4w^2 + 3w - 7 = 0$?

This factors to $(4w + 7)(w - 1) = 0$ with roots of $-\frac{7}{4}$ and 1.

13. What value(s) of m satisfy $\frac{-6m+1}{3m+5} = \frac{12m-9}{-6m+5}$?

Cross-multiplying gives $36m^2 - 36m + 5 = 36m^2 + 33m - 45$, then $50 = 69m$, giving $m = \frac{50}{69}$.

14. If $k(j, h) = 9jh - 8j - 7h^2 - \frac{j}{h}$, evaluate $k(4, 2)$.

$k(4,2) = 9 \cdot 4 \cdot 2 - 8 \cdot 4 - 7 \cdot 2^2 - \frac{4}{2} = (18 - 8 - 7) \cdot 4 - 2 = 3 \cdot 4 - 2 = 12 - 2 = 10$

15. When Mr. Brown put an equation of the form $0 = x^2 + Bx + C$ on the board, Sam miscopied the value of B and got roots of -3 and 4. On the same problem, Anthea miscopied the value of C and got roots of 0 and 4. What were the roots of the original problem?

Sam got C right, so it must have been $-3 \cdot 4 = -12$. Anthea got B right, so it must have been $-(0 + 4) = -4$. Thus, the original equation must have been $0 = x^2 - 4x - 12 = (x - 6)(x + 2)$, with roots of 6 & -2.

16. What are the coordinates, in the form (x, y), of the left-most x-intercept of the parabola $y = -x^2 - 7x + 9$?

The quadratic formula gives $\frac{-(-7)\pm\sqrt{(-7)^2-4(-1)9}}{2(-1)} = \frac{7\pm\sqrt{49+36}}{-2} = \frac{-7\pm\sqrt{85}}{2}$, for an answer of $\left(\frac{-7-\sqrt{85}}{2}, 0\right)$.

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17. What is the shortest distance from the point $(2, 4)$ to the line $x - 3y = 4$?

Rewriting the line as $1x - 3y - 4 = 0$, the shortest distance becomes $\frac{|1\cdot 2 - 3\cdot 4 - 4|}{\sqrt{1^2+3^2}} = \frac{|2-12-4|}{\sqrt{10}} = \frac{14}{\sqrt{10}} = \frac{14\sqrt{10}}{10} = \frac{7\sqrt{10}}{5}$.

18. If I drive 501 kilometers in 25 hours, then 564 kilometers in 46 hours, what is my average speed in kilometers per hour?

The total distance traveled is $501 + 564 = 1065$ in $25 + 46 = 71$ hours, for an average speed of $\frac{1065}{71} = 15$.

19. If Pete could roof the shed in 16 hours and Tom could roof it in 24 hours, how many minutes would it take them if they worked together?

Their speeds are $\frac{1}{16}$ and $\frac{1}{24}$ of the roof per hour, for a combined speed of $\frac{3+2}{48} = \frac{5}{48}$ of the roof per hour, so doing one roof would take $\frac{48}{5}$ hours, which is $\frac{48}{5} \cdot 60 = 48 \cdot 12 = 24^2 = 576$ minutes.

20. What is the area of a right triangle with a hypotenuse measuring 15 m and a leg measuring 12 m?

A 15-12-x triangle is three times as large as a 5-4-y triangle, so $x = 3 \cdot 3 = 9$, for an answer of $A = \frac{1}{2} \cdot 9 \cdot 12 = 9 \cdot 6 = 54$.

21. What is the name for a triangle with exactly two congruent sides?

You just need to memorize that this is an “isosceles” triangle.

22. A triangle has two sides measuring 84 m and 26 m. If the third side measures c meters, where c is a whole number, what is the largest possible value of c?

If the two given sides form nearly a straight angle, the third side could be as large as $84 + 26 - 1 = 109$.

23. What is the circumference, in meters, of a circle circumscribed about a square with a perimeter of 16 m?

If the perimeter is 16, the length of a side is $\frac{16}{4} = 4$, the diagonal is $4\sqrt{2}$, so the radius of the circumscribing circle is $\frac{4\sqrt{2}}{2} = 2\sqrt{2}$, for an answer of $C = 2\pi r = 2\pi \cdot 2\sqrt{2} = 4\pi\sqrt{2}$.

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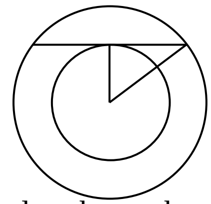
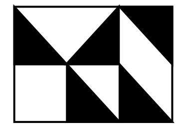


24. Two concentric circles contain an annular area of 48π m^2. What is the length, in meters, of a chord of the larger circle that is tangent to the smaller circle?

The figure to the right shows how $A = 48\pi = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$. It also shows that half the length of the chord is $\sqrt{R^2 - r^2}$, making our answer $2\sqrt{48} = 4\sqrt{12} = 8\sqrt{3}$.

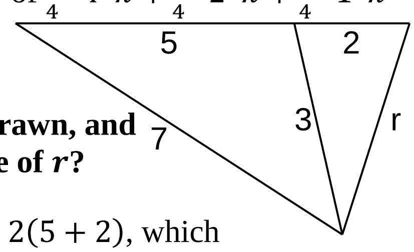


25. What fraction of the 2x3 rectangle to the right is shaded? Assume that shading occurs at $45°$ angles.

If the lower left square were divided into two white triangles, there would be 12 congruent triangles, five of which would be shaded, for an answer of $\frac{5}{12}$.

26. A goat is tied to an exterior corner of an enclosed rectangular shelter measuring 2 m by 3 m. If the length of the rope is 4 m, what is the area, in square meters, that the goat can graze?

The goat can graze $\frac{3}{4}$ of a circle with a radius of 4 m, $\frac{1}{4}$ of a circle with a radius of $4 - 2 = 2$ m, and $\frac{1}{4}$ of a circle with a radius of $4 - 3 = 1$, for an answer of $\frac{3}{4} \cdot 4^2\pi + \frac{1}{4} \cdot 2^2\pi + \frac{1}{4} \cdot 1^2\pi = 12\pi + \pi + \frac{\pi}{4} = \frac{53\pi}{4}$.

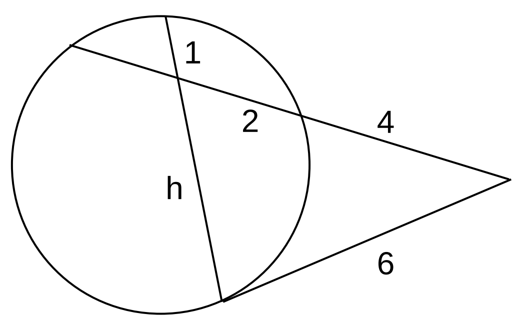


27. The figure to the right shows a triangle with one cevian drawn, and all segment lengths are given in meters. What is the value of r?

Stewart's Theorem gives $r^2 \cdot 5 + 7^2 \cdot 2 = 3^2 \cdot (5 + 2) + 5 \cdot 2(5 + 2)$, which becomes $5r^2 + 98 = 63 + 70$, then $5r^2 = 35$, so that $r^2 = 7$, giving $r = \sqrt{7}$.

28. A triangle has sides measuring 6 m, 9 m, and 5 m. What is the length, in meters, of the altitude to the shortest side?

The area of the triangle is $\sqrt{10 \cdot 1 \cdot 5 \cdot 4} = 10\sqrt{2} = \frac{1}{2} \cdot 5 \cdot h$, giving $h = 10\sqrt{2} \cdot \frac{2}{5} = 4\sqrt{2}$.



29. The figure to the right includes a circle, a chord, a tangent, and a secant, with most line segments labeled in meters. What is the value of h?

The secant and tangent allow us to write $6^2 = 4(4 + 2 + x)$ to get the missing length in the upper left. This becomes $36 = 24 + 4x$, then $12 = 4x$, giving $3 = x$. Now the chords allow us to write $3 \cdot 2 = 1 \cdot h$, which gives $6 = h$.

30. If $c(d) = 3d^2 - 1$ and $f(g) = 4 + 5g$, evaluate $f^{-1}(c(10))$.

$c(10) = 3 \cdot 10^2 - 1 = 3 \cdot 100 - 1 = 300 - 1 = 299$, so the question is what value of g will result in $f = 299 = 4 + 5g$, so $295 = 5g$ and $59 = g$.

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31. What is the product of the roots of the polynomial $4h^6 - 2h^7 + 4h^9 + 6h^8 + 6 = 3$?

This polynomial starts with $\mathbf{4h^9}$ and ends with $\mathbf{+3}$, for an answer of $\frac{(-1)^9 3}{4} = -\frac{3}{4}$.

32. Express the range of $q(r) = 2r^2 + 4r - 3$ in interval notation. Assume the domain and range are both subsets of the real numbers.

This is an upward-pointing parabola, so the range will be from the vertex to infinity. The axis of symmetry is $x = -\frac{4}{2\cdot 2} = -\frac{4}{4} = -1$, so the vertex is at $y = 2(-1)^2 + 4(-1) - 3 = 2 - 4 - 3 = -5$, for an answer of $[-5, \infty)$.

33. Express the base 6 numeral 2301_6 as a base 10 numeral.

From right to left, the digits in base 6 represent 1s, 6s, 36s, and 216s. $2 \cdot 216 + 3 \cdot 36 + 1 \cdot 1 = 432 + 108 + 1 = 541$

34. Express the base 10 numeral 6984_{10} as a base 9 numeral.

In base 9, the digits from right to left represent 1s, 9s, 81s, 729s, and 6561s. There is 1 6561, leaving 423. There are 0 729s. There are 5 81s, leaving 18, which is 2 9s, for an answer of 10,520.

35. What is the units digit when 697^{231} is evaluated?

This is the same as if it were just 7^{231}. Powers of 7 follow the pattern 7, 9, 3, 1, then repeats. 228 is a multiple of 4, so our answer is 3.

36. What is the sum of the positive integer factors of 960?

$960 = 10 \cdot 96 = 2 \cdot 5 \cdot 2^5 \cdot 3 = 2^6 \cdot 3 \cdot 5$, so the sum of the factors will be $(1 + 2 + 4 + 8 + 16 + 32 + 64)(1 + 3)(1 + 5) = 127 \cdot 4 \cdot 6 = 127 \cdot 24 = 3048$.

37. What is the missing term of the sequence $1, 8, 23, 43, 65, 86,$ __, …?

The differences are 7, 15, 20, 22, 21, … It's still not clear; let's do it again, getting 8, 5, 2, -1. Aha! The next term of that sequence will be $-1 - 3 = -4$, so the next difference will be $21 - 4 = 17$, making the answer $86 + 17 = 103$.

38. What is the missing term of the harmonic sequence 4, 3, ___, …?

A harmonic sequence is a constant divided by the terms of an arithmetic sequence. One expression for this sequence could be $\frac{12}{3}, \frac{12}{4}$, ___, so the next term would be $\frac{12}{5}$.

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39. What is the missing term of the sequence 1, 2, 1, 3, 2, 4, 3, 5, 5, 6, 8, __, …?

After trying many other things, consider that this may be two or more interspersed sequences: 1, 1, 2, 3, 5, 8, … and 2, 3, 4, 5, 6, __, … The former is the Fibonacci Sequence, and the latter is linear, making the answer 7.

40. What is the sum of the counting numbers less than 30?

Using outer pairs, the numbers from 1 to 29 add up to $(1 + 29) \cdot \frac{29}{2} = \frac{30 \cdot 29}{2} = 15 \cdot 29 = 435$.

41. What is the 9th term of a geometric sequence with first term 8 and common ratio 3?

$8 \cdot 3^8 = 8 \cdot 81 \cdot 81 = 8 \cdot 6{,}561 = 52{,}488$

42. When two cards are drawn from a standard 52-card deck, what is the probability that they are of different ranks?

The first card is irrelevant. The probability that the second card matches the first is $\frac{3}{51} = \frac{1}{17}$, for an answer of $1 - \frac{1}{17} = \frac{16}{17}$.

43. A bag contains 8 red marbles and 6 blue marbles. A trusted friend draws two marbles from the bag, looks at them, and tells you they are the same color. What is the probability that they are both blue?

You might think there are $14c2 = \frac{14 \cdot 13}{2} = 7 \cdot 13 = 91$ ways the marbles could have been drawn, but that includes cases where the marbles were not the same color, which your friend has assured you did not happen. In reality, there are $8c2 + 6c2 = \frac{8 \cdot 7}{2} + \frac{6 \cdot 5}{2} = 4 \cdot 7 + 3 \cdot 5 = 28 + 15 = 43$ ways the marbles could have been drawn, and 15 of them are double blue, for an answer of $\frac{15}{43}$.

44. I have three different red books, two different blue books, and four different green books. I'd like to put them on a bookshelf so that same-colored books are next to one another. In how many ways can I do this?

There are three colors that can be arranged in $3! = 6$ ways. Within each color, there are $3! = 6$, $2! = 2$, and $4! = 24$ ways, for an answer of $6 \cdot 6 \cdot 2 \cdot 24 = 12^3 = 1728$.

45. In a school with 48 boys and 19 girls, 39 students are failing, but 13 girls are passing. How many boys are passing?

This is a Venn diagram problem for the intersection of gender with passingness. There are $48 + 19 = 67$ total students, of which 48 are boys and $67 - 39 = 28$ are passing. The 48 boys and the 13 passing girls imply there are $67 - 48 - 13 = 67 - 61 = 6$ girls failing, meaning that $39 - 6 = 33$ boys are failing, leaving $48 - 33 = 15$ passing boys.

46. My fashion teacher has instructed me to bring four socks to school today, consisting of exactly three different colors (exactly two of the socks will be the same color). I forgot to set them aside last night, and because I share a room with my younger sister, I need to grab the socks in the dark before I slip out of my room. Fortunately, I know that my sock drawer contains 1 purple, 4 blue, 6 green, 9 yellow, 4 orange, and 9 red socks. What is the smallest number of socks I can grab in the dark to be sure that I can select four that meet the requirements of my fashion assignment once I get into the light?

I need to get three different colors, so the worst thing that could happen is drawing 9 yellow and 9 socks, for an answer of $9 + 9 + 1 = 19$. We need to check that 19 is also sufficient to be sure we get at two matching socks, which it is because there are only six different colors, so if we draw 7 socks we'll be sure to have two matching socks. Our answer of 19 is well above this.

47. Evaluate: $\begin{vmatrix} \mathbf{5} & \mathbf{6} & \mathbf{0} \\ \mathbf{3} & \mathbf{-2} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{9} \end{vmatrix}$

$5(-2 \cdot 9 - 1 \cdot 3) - 3(6 \cdot 9 - 0 \cdot 3) = 5(-21) - 3(54) = -105 - 162 = -267$

48. Set D is {7, 6, 3, 64, 9}, Set E is {50, 70, 9, 2, 4, 1, 5}, and Set F is {6, 67, 7, 3, 5, 9, 8}. Write the set $(F \cap E') \cup D$.

We're looking for all of D, plus anything in F but not E, which is $\{7, 6, 3, 64, 9, 67, 8\}$.

49. Set Q is {-9, -6, 7, 2, 5, -8}, and Set P is {3, 1, 4, 2, 7}. Set N is the largest set that is a subset of both Q and P. Set M is the smallest set that is a superset of both Q and P. What is the product of the number of elements in Set N and the number of elements in Set M?

N is $\{7, 2\}$, which is 2 elements. M is {-9, -6, 7, 2, 5, -8, 3, 1, 4}, which is 9 elements, so that the answer is $2 \cdot 9 = 18$.

50. Evaluate: $\mathbf{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$

Let $c = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}} = 2 + \frac{1}{c}$. Multiplying both sides by c gives $c^2 = 2c + 1$, then $c^2 - 2c - 1 = 0$. The Quadratic Equation gives $c = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1(-1)}}{2 \cdot 1} = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$. Because the original expression is clearly positive, the answer is $1 + \sqrt{2}$.

51. If $\sec k = -\frac{5}{4}$, what is the largest possible value of $\cot k$?

$\cos k = -\frac{4}{5}$, so $\sin k = \pm\frac{3}{5}$, for an answer of $\frac{4}{3}$.

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52. If h and j are angles in the first quadrant, $\sin h = \frac{3}{5}$, and $\sin j = \frac{5}{13}$, evaluate $\sin(h + j)$.

$\sin h + j = \sin h \cos j + \sin j \cos h = \frac{3}{5} \cdot \frac{12}{13} + \frac{5}{13} \cdot \frac{4}{5} = \frac{36+20}{65} = \frac{56}{65}$

53. If $\cos m = -\frac{1}{3}$ and m is in the third quadrant, what values can $\sin \frac{m}{2}$ take?

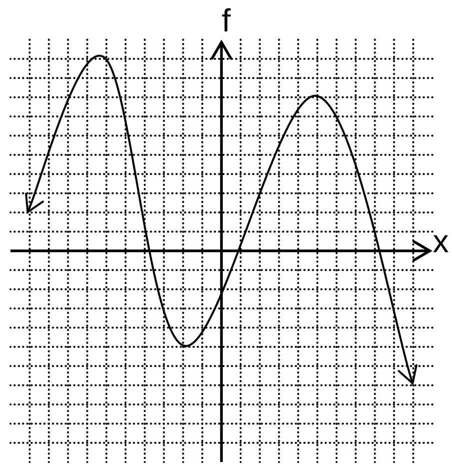
Because $\cos 2b = 1 - 2\sin^2 b$, we can write $\sin b = \pm\sqrt{\frac{1-\cos 2b}{2}}$, so $\sin \frac{m}{2} = \pm\sqrt{\frac{1-\left(-\frac{1}{3}\right)}{2}} = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$.

54. Evaluate: $\lim\limits_{n \to 0} \frac{\ln(e-3n)-1}{n}$

This is -3 times the definition of the derivative for $y = \ln x$ at $x = e$. The derivative of $y = \ln x$ is $\frac{1}{x}$, so our answer is $-\frac{3}{e}$.

55. Approximate $\sqrt{9.2}$ using a first-order differential for $y = \sqrt{x}$ about $x = 9$.

$y(9) = \sqrt{9} = 3$, and $y'(x) = \frac{1}{2\sqrt{x}}$, so $y'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2 \cdot 3} = \frac{1}{6}$. Thus, $y(9.2) \approx 3 + \frac{1}{6} \cdot .2 = 3 + \frac{1}{6} \cdot \frac{1}{5} = 3\frac{1}{30} = \frac{91}{30}$.



56. Given the graph of f to the right (a fourth-degree polynomial), on what interval(s) is f' decreasing as x increases? Asusme that all points of interest occur at the nearest integer value of x. Note the unit grid…

If f' is decreasing, then f'' should be negative, which means the graph is concave down. This happens on the left and right, for an answer of $(-\infty, -4) \cup (2, \infty)$.

Harder Problems

57. Evaluate: $68^3 - 32^3$

$68^3 - 32^3 = (68 - 32)(68^2 + 68 \cdot 32 + 32^2) = 36((68 + 32)^2 - 68 \cdot 32) = 36\left(100^2 - (50^2 - 18^2)\right) = 36(10{,}000 - 2500 + 324) = 36 \cdot 7824 = 281{,}664$

58. Your piggy bank contains 12 half-dollars, 30 quarters, 98 dimes, 4 nickels, and 76 pennies. As a decimal, how many dollars is it all worth?

$12 \cdot .5 + 30 \cdot .25 + 98 \cdot .1 + 4 \cdot .05 + 76 \cdot .01 = 6 + 7.5 + 9.8 + .2 + .76 = 24.26$

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59. You buy two dozen donuts for \$7.80, and are charged 10% tax on that price. If the baker throws an extra donut into each dozen, how many cents are you paying per donut?

$780 \cdot \frac{1.1}{26} = 30 \cdot 1.1 = 33$

60. Express in simplest radical form: $\sqrt{1920}$

$\sqrt{1920} = 2\sqrt{480} = 4\sqrt{120} = 8\sqrt{30}$

61. Six years ago, JT was three times as old as Wyatt. Five years from now, he'll be twice as old as Wyatt. How old is JT now?

We can write $j - 6 = 3(w - 6)$ and $j + 5 = 2(w + 5)$, which become $j - 6 = 3w - 18$ and $j + 5 = 2w + 10$. Subtracting these gives $11 = -w + 28$, giving $w = 17$, so that $j = 3 \cdot 17 - 18 + 6 = 51 - 12 = 39$.

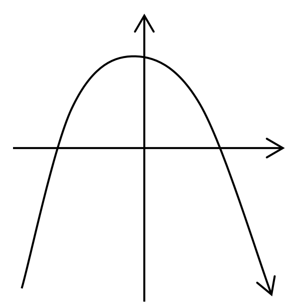
62. Expand and combine like terms: $(a + b - c)(2a - b + 3c)$

$(a + b - c)(2a - b + 3c) = 2a^2 - ab + 3ac + 2ab - b^2 + 3bc - 2ac + bc - 3c^2 = 2a^2 + ab + ac - b^2 + 4bc - 3c^2$

63. What is the point of intersection, in the form (x, y), of the lines $2x + 6y = 2$ and $y = 6x - 8$?

Substituting gives $2x + 6(6x - 8) = 2$, which becomes $2x + 36x - 48 = 2$, then $38x = 50$, so that $x = \frac{50}{38} = \frac{25}{19}$, which makes $y = 6 \cdot \frac{25}{19} - 8 = \frac{150}{19} - \frac{152}{19} = -\frac{2}{19}$, for an answer of $\left(\frac{25}{19}, -\frac{2}{19}\right)$.

64. Ignoring scaling, which of the following might be the equation of the parabola shown?



A. $y = +2x^2 + 3x + 97$
B. $y = -2x^2 + 3x + 97$
C. $y = +2x^2 + 3x - 97$
D. $y = -2x^2 + 3x - 97$
E. $y = +2x^2 - 3x + 97$
F. $y = -2x^2 - 3x + 97$
G. $y = +2x^2 - 3x - 97$
H. $y = -2x^2 - 3x - 97$

The parabola points down, so the answer needs to be B, D, F, or H. The axis of symmetry is at a negative x-value, which restricts us to F or H. There are two x-intercepts, thus a positive discriminant, making our answer F.

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65. When the math team goes out for dessert after a contest, they decide to split a Too Huge Chocolate Cake. If there had been one more person, each person would have paid 75 cents less. If there had been two more people than originally, each person would have paid $1.35 less than they actually did. How many dollars does the cake cost?

We can write $C = ne = (n+1)(e-.75) = (n+2)(e-1.35)$. This is three equations in two unknowns, so hopefully there's some dependence. Examining at the first pair, we can write $e - .75n = .75$, and the last pair gives $2e - 1.35n = 2.70$. Doubling the first of these and subtracting gives $.15n = 1.2$, so that $n = \frac{1.2}{.15} = \frac{120}{15} = 8$, giving $e = .75 + .75 \cdot 8 = 9 \cdot .75 = 6.75$, for an answer of $8 \cdot 6.75 = 54$.

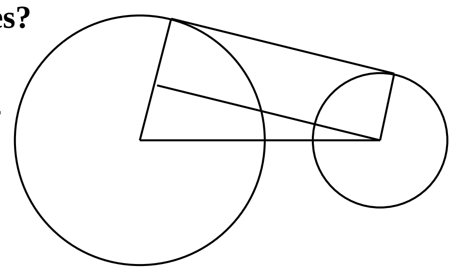
66. What value(s) of b can be part of a solution to the system of equations $b + 2c + 3d + 4f = 25$, $2b + c - 3d + 2f = -10$, and $b - c + d - 2f = -14$?

With three equations and four unknowns, there'd better be some dependence going on… Cancelling c gives $-3b + 9d = 45$ which also cancelled f and is really $-b + 3d = 15$ and $3b - 2d = -24$. Now canceling b gives $7d = 21$ which becomes $d = 3$, and gives $b = 3 \cdot 3 - 15 = 9 - 15 = -6$.

67. What is the most specific name that could be given to a quadrilateral with sides measuring 1 m, 2 m, 3 m, and 4 m (not necessarily in that order)?

We're clearly not going to have anything with congruent sides (rhombus, parallelogram, kite), but perhaps we could have something with parallel sides. Could the 1 and 2 be parallel? No, because their lengths differ by 1, which is the same as the difference of the sides, 3 and 4. Could the 1 and 3 be parallel? No, for the same reason. How about the 1 and 4? Yes, this is clearly possible. Thus, this quadrilateral can be a **trapezoid**, which is sufficient for 1 point. There is a **bonus point** available for teams that applied at least one of the following adjectives: convex, concave, irregular. Your team could get **two bonus points** if they applied irregular and either convex or concave.

68. Two circles have radii of 44 m and 50 m, and their centers are 110 m apart. What is the length, in meters, of an external tangent of those circles?



The desired tangent is the same length as the leg of a right triangle with a hypotenuse of 110 and another leg of $50 - 44 = 6$, for an answer of $\sqrt{110^2 - 6^2} = 2\sqrt{55^2 - 3^2} = 2\sqrt{3025 - 9} = 2\sqrt{3016} = 4\sqrt{754}$.

69. What is the first time after 9:41 AM where the minute hand of a standard 12-hour analog clock is exactly opposite the hour hand? Answer in the form HH:MM:SS to the nearest second, including AM or PM.

At 9:41, the minute hand is about to catch up to the hour hand before 9:50, so the first opposition will be in the 10:20 timeframe. After 10 AM, the angle of the minute hand is $6m$, while that of the hour hand is $300+\frac{m}{2}$, so we can write $300+\frac{m}{2}=6m+180$, which becomes $120=\frac{11m}{2}$, giving $m=\frac{240}{11}=21R9$, so that the desired time is 10:21 and $\frac{9}{11}\cdot 60=\frac{540}{11}\sim 49$ seconds.

70. The faces of a solid ten-inch cube of white plastic are each painted a different non-white color, then the cube is cut into two-inch cubes. How many of these smaller cubes have faces of at least three different colors (potentially including white)?

Each of the $\left(\frac{10}{2}\right)^3=5^3=125$ cubes has at least one white face, so we're interested in those with at least two other colors, which is the vertices and the edges, for an answer of $8+12\cdot 3=8+36=44$.

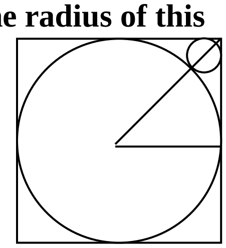
71. What is the area of an isosceles triangle with sides measuring 8 m and 3 m?

Using Heron's Formula, we get $A=\sqrt{\frac{19}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{13}{2}}=\frac{3\sqrt{247}}{4}$.

72. What is the length, in meters, of the angle bisector of the largest angle in a triangle with sides measuring 7, 4, and 9 m?

The angle bisector divides the 9 side into $\frac{4}{11}\cdot 9=\frac{36}{11}$ and $\frac{7}{11}\cdot 9=\frac{63}{11}$, so that Stewart's Theorem gives $7^2\cdot\frac{36}{11}+4^2\cdot\frac{63}{11}=9t^2+9\cdot\frac{36}{11}\cdot\frac{63}{11}$. Multiplying by $\frac{121}{9}$ gives $7^2\cdot 11\cdot 4+4^2\cdot 7\cdot 11=121t^2+36\cdot 63$. This becomes $2156+1232-2268=121t^2$, then $t^2=\frac{1120}{121}$, giving $t=\frac{\sqrt{1120}}{11}=\frac{2\sqrt{280}}{11}=\frac{4\sqrt{70}}{11}$.

73. A circular rug with a radius of 3 meters is pushed into the corner of a rectangular room so that it touches two walls. A smaller circular rug needs to be designed to fit into the corner so that it touches both walls and the large rug. What should the radius of this smaller rug be, in meters?



The figure to the right allows us to write $3+r+r\sqrt{2}=3\sqrt{2}$, which becomes $(1+\sqrt{2})r=3\sqrt{2}-3$, giving $r=\frac{3\sqrt{2}-3}{1+\sqrt{2}}=\frac{(3\sqrt{2}-3)(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}=\frac{6\sqrt{2}-9}{1-2}=9-6\sqrt{2}$.

74. I invest a million dollars in an account with a 4% interest rate that compounds quarterly. How much money, to the nearest hundredth of a dollar (cent), will be in the account after two years?

$1{,}000{,}000 \cdot 1.01^8 = 1{,}000{,}000 \cdot 1.0828567056280801 = 1{,}082{,}856.7056280801$, for an answer of 1,082,856.71.

75. In the hyperbola with equation $\frac{(y-8)^2}{5} - \frac{(x+2)^2}{16} = 1$, what is the length of a latus rectum?

The latus rectum length depends only on the shape of the conic section, not on its position, so we can simplify our work by playing with $\frac{y^2}{5} - \frac{x^2}{16} = 1$. This has vertices of $(0, \pm\sqrt{5})$ and foci of $(0, \pm\sqrt{5+16}) = (0, \pm\sqrt{21})$. The latus rectum is the line segment through a focus parallel to the directrix, which in this case is horizontal. So, what value of x corresponds to $y = \pm\sqrt{21}$? $\frac{21}{5} - \frac{x^2}{16} = 1$ becomes $\frac{x^2}{16} = \frac{21}{5} - 1 = \frac{16}{5}$, so $x^2 = \frac{16^2}{5}$, giving $x = \pm\frac{16}{\sqrt{5}} = \pm\frac{16\sqrt{5}}{5}$, for an answer of $2 \cdot \frac{16\sqrt{5}}{5} = \frac{32\sqrt{5}}{5}$.

76. When Mr. E writes an equation of the form $x^3 + Bx^2 + Cx + D = 0$ on the board, Isabella, Sophia, and Mateo miscopy two values each, only getting B, C, and D correct, respectively. When they solve their respective equations, Isabella gets roots of 1, 2, and -10, Sophia gets roots of 1, -2, and 52, and Mateo gets roots of 2, 2, and -18. What were the roots of the original equation?

Isabella got B right, so B must have been $-(1+2-10) = 7$. Sophia got C right, so it must have been $1(-2) + 1 \cdot 52 + (-2)52 = -2 + 52 - 104 = -54$. Mateo got D right, so it must have been $-2 \cdot 2(-18) = 72$. The original equation must have been $x^3 + 7x^2 - 54x + 72$. Testing the possible rational roots, 1 doesn't work but 2 does, so the equation factors to $(x-2)(x^2+9x-36) = (x-2)(x-3)(x+12)$, for an answer of 2, 3, and -12.

77. Express the product of 231_5 and 421_5 as a base 5 numeral.

We can work entirely in base 5, which will save us the effort of converting to and from base ten; the trick is that we need to carry 5s, not 10s. $1_5 \times 421_5 = 421_5$, $30_5 \times 421_5 = 23{,}130_5$, and $200_5 \times 421_5 = 134{,}200_5$, for an answer of $421_5 + 23{,}130_5 + 134{,}200_5 = 213{,}301_5$ (again, carrying 5s).

78. What is the least common multiple of 396 and 264?

$396 = 6 \cdot 66 = 2^2 \cdot 3^2 \cdot 11$ and $264 = 4 \cdot 66$, oh, that's enough; we don't have to do the whole prime factorization. The least common multiple of $6 \cdot 66$ and $4 \cdot 66$ will be $12 \cdot 66 = 792$.

79. How many palindromes between 3519 and 8098 are multiples of three and contain the digit 5?

Palindromes in the given range with at least one 5 are 3553, 4554, 5005, 5115, 5225, 5335, 5445, 5555, 5665, 5775, 5885, 5995, 6556, and 7557. Numbers divisible by 3 have digits that sum to 3, which is 4554, 5115, 5445, 5775, and 7557, or an answer of 5.

80. What is the largest number less than 1000 that leaves a remainder of six when divided by eight and a remainder of three when divided by nine?

Numbers that leave a remainder of 3 when divided by 9 are 3, 12, 21, 30, 39, 48, 57, 66, 75, … Of these, 30 has a remainder of 6 when divided by 8. After this, it'll be every 72 numbers, because that's the least common multiple of 8 and 9. So we're looking for a number that leaves a remainder of 30 when divided by 72. $\frac{1000}{72} = 13R64$, so our answer is $1000 - 64 + 30 = 1000 - 34 = 966$.

81. How many positive four-digit integers contain exactly one 3 and at least one 4?

There are $9 \cdot 9 \cdot 9 = 729$ four-digit numbers that start with their one 3, and $3 \cdot 8 \cdot 9 \cdot 9 = 1944$ with their 3 elsewhere, for a total of $729 + 1944 = 2673$ four-digit numbers with exactly one 3. Of these, $8 \cdot 8 \cdot 8 + 3 \cdot 7 \cdot 8 \cdot 8 = 512 + 1344 = 1856$ don't contain a 4, so our answer is $2673 - 1856 = 817$.

82. What is the missing term of the sequence 2, 3, 8, 63, ___, …?

There aren't many terms, and it's not obviously arithmetic, geometric, or harmonic. What if it were recursive? How might 63 be based on 8? Well, $8^2 = 64$, so perhaps the formula is $f_n = f_{n-1}^2 - 1$. In fact, this works for every term after the first, so our answer will be $63^2 - 1 = 3968$.

83. Sequence A is arithmetic with first term 8 and common difference 7. Sequence B is geometric with first term 1 and common ratio 2. What is the sum of the numbers less than 10,000 that are in both sequences?

The nth term of sequence A is $1 + 7n$, and the mth term of sequence B is $\frac{1}{2} \cdot 2^m$. There are fewer of B to think about, so we'll start there. The terms are 2, 4, 8, 16, 32, 64, … To see if they're of the form $1 + 7n$, we'll subtract 1 (1, 3, 7, 15, 31, 63, …) and see if they're divisible by 7. Obviously, this is just 7 and 63, which makes us think it'll be every three. Let's check 128 (127), 256 (255), and 512 (511). Again, it's no, no, yes, so our answer will be $8 + 64 + 512 + 4096 = 4680$.

84. What is the sum of the positive integers less than 100 that are not multiples of 8?

The sum of 1-99 is $(1 + 99) \cdot \frac{99}{2} = 100 \cdot \frac{99}{2} = 50 \cdot 99 = 4950$. The sum of the first 12 multiples of 8 (8-96) is $8(1 + 12) \cdot \frac{12}{2} = 8 \cdot 13 \cdot 6 = 104 \cdot 6 = 624$, giving an answer of $4950 - 624 = 4326$.

85. A recursive sequence is defined with first term $f_1 = 54$ and subsequent terms $f_n = \frac{f_{n-1}}{\sqrt{3}} + \sqrt{3}$. Evaluate f_8, writing your answer in the form $b + c\sqrt{3}$ where b and c are real numbers.

$f_2 = \frac{54}{\sqrt{3}} + \sqrt{3}$, $f_3 = \frac{54}{3} + 1 + \sqrt{3} = 19 + \sqrt{3}$, $f_4 = \frac{19}{\sqrt{3}} + 1 + \sqrt{3} = \frac{22}{\sqrt{3}} + 1$, $f_5 = \frac{22}{3} + \frac{1}{\sqrt{3}} + \sqrt{3} = \frac{22}{3} + \frac{4\sqrt{3}}{3}$, $f_6 = \frac{22}{3\sqrt{3}} + \frac{4}{3} + \sqrt{3} = \frac{31}{3\sqrt{3}} + \frac{4}{3}$, $f_7 = \frac{31}{9} + \frac{4}{3\sqrt{3}} + \sqrt{3} = \frac{31}{9} + \frac{13}{3\sqrt{3}}$, and $f_8 = \frac{31}{9\sqrt{3}} + \frac{13}{9} + \sqrt{3} = \frac{58}{9\sqrt{3}} + \frac{13}{9} = \frac{58\sqrt{3}}{27} + \frac{13}{9}$.

86. On tomorrow's test, the probabilities that Katie, Wily, and Emily get A's are .9, .8, and .7, respectively. What is the probability, as a decimal, that exactly two of them get A's?

The total probability will be $.9 \cdot .8 \cdot .3 + .9 \cdot .2 \cdot .7 + .1 \cdot .8 \cdot .7 = .216 + .126 + .056 = .398$.

87. Jim & Julie play a game in which they take turns rolling a standard six-sided die. The person rolling the die wins if they roll a 1 or 2, or if they roll a number higher than the other person just rolled. What is the probability that the second player wins on their first turn?

We're looking for the probability that the first player does not win on their turn, and that the second player does. This could happen in the following ways: 3(1, 2, 4, 5, or 6), 4(1, 2, 5, or 6), 5(1, 2, or 6), or 6 (1 or 2). This is $5 + 4 + 3 + 2 = 14$ of the $6 \cdot 6 = 36$ total ways the two could roll the die, for an answer of $\frac{14}{36} = \frac{7}{18}$.

88. On the game show Let's Make a Heel, there are one million doors you can choose from, one of which has the cobbler's tools you desire behind them, and the rest of which have nothing behind them. On the show, you get to choose a door, but before you open it, the show's host will open 999,998 doors that she knows to have nothing behind them. At that point, you have the opportunity to open either the door you originally chose or the other remaining unopened door. If you choose to open the door you did not originally choose, what is the probability that your coveted cobbler's tools are behind it?

Initially, the chance I pick the right door is $\frac{1}{1{,}000{,}000}$. The host can *always* open 999,998 bad doors, so I don't learn anything about the door I picked; it still has a $\frac{1}{1{,}000{,}000}$ probability of being the right door. That means that the one remaining other door has a $\frac{999{,}999}{1{,}000{,}000}$ chance of being the right door.

2018 Team Scramble
Solutions

89. What is the equation, in the form $y = f(x) = g(z)$, of the line through the points $(8, -8, -2)$ and $(5, 1, -1)$?

The line travels in the direction $< -3,9,1 >$, making its equation $\frac{x-8}{-3} = \frac{y+8}{9} = \frac{z+2}{1}$. Multiplying by 9 gives $y + 8 = 24 - 3x = 9z + 18$, for an answer of $y = 16 - 3x = 9z + 10$.

90. In the data set {9, 48, 5, 56, 9, x, y}, where x and y are counting numbers, the unique mode is greater than the median, which is greater than the mean. What is the largest possible value of $x + y$?

In order, the known elements are $\{5, 9, 9, 48, 56\}$, so the median is between 9 and 48 inclusive, the mode is either 9 or $x = y =$ another known element, and the mean is $\frac{23+104+x+y}{7} = \frac{127+x+y}{7} = \frac{127}{7} + \frac{x+y}{7} = 18\frac{1}{7} + \frac{x+y}{7}$. If the mode is greater than the median, the mode cannot be 9, so $x = y =$ another known element. For our goal of a large $x = y$, we'd love to pick 56, in which case the mode would be 56, the median would be 48, and the mean would be $\frac{127+112}{7} = \frac{239}{7} = 34\frac{1}{9}$, making our answer $56 + 56 = 112$.

91. Many sets of counting numbers have the properties that their range is less than their unique mode and their mean is greater than their median. Of the many such sets, some of them have the smallest number of elements for such a set. Of these sets, one of them has the smallest mean. Write that set as your answer.

Whatever the set is, it will have at least two elements that are the mode. For the mean to be different from the median, there will have to be elements other than the mode. For the mean to be greater than the median, there will need to be at least one element greater than the mode. The smallest such set would be mode, mode, X; for its range to be less than the mode, either X should be close to the mode or the mode should be large. Thus, it appears that the answer can be a three-element set. In order for the mean to be small, the mode will need to be fairly small and the X will need to be close to the mode. The smallest the range could be is 1, so the smallest the mode can be is 2, for an answer of $\{2, 2, 3\}$.

92. Set B is the set of all positive three-digit multiples of 5, and Set C is the set of integers between 50 and 500 (inclusive) with at least one digit that is a 5. How many elements are in the set $B \cup C$?

$B \cup C$ contains all of B and all of C. B has $\frac{900}{5} = 180$ elements. The elements of C that are NOT in B are the 50s (10), 65-95 (4), the 150s (8), the 250s (8), the 350s (8), and the 450s (8), for a total of $10 + 4 + 8 + 8 + 8 + 8 = 14 + 32 = 46$, for an answer of 226.

93. In a five-digit counting number, the first digit is twice the second digit, the third digit is three times the fourth digit, and the average of the digits is 6. What is the smallest number satisfying these constraints?

The first two digits could be 21, 42, 63, or 84. The third & fourth could be 31, 62, or 93. For the average to be 6, the sum must be $5 \cdot 6 = 30$. The largest the fifth digit could be is 9, so the smallest the sum of the other four could be is $30 - 9 = 21$. $9 + 3 = 12$, so the smallest the first two digits could sum to is $21 - 12 = 9$, so the smallest number is 63939.

94. A Coup deck contains three each of Assassins, Captains, Contessas, Dukes, and Ambassadors. After a particular shuffle, all three Assassins are adjacent, no Captains are, exactly two Contessas are, no Dukes are, and exactly two Ambassadors are. In addition, there is at least one Captain that is adjacent to two Dukes, and at least one Ambassador that is adjacent to two Contessas. Furthermore, the top card in the deck is a Duke and the bottom card is an Ambassador. Finally, all of the Dukes are higher in the deck than all of the Assassins. Given all this, how many distinguishable arrangements of cards are possible?

We can write AsAsAs, Ca-Ca-Ca, CoCo-Co, D-D-D, and AmAm-Am Two are then modified to become DCaD-D and CoCoAmCo-AmAm. We can then write *TOP*DCaD-D or *TOP**D-DCaD and CoCoAmCo-AmAm*BOT*. At this point, there is definitely the TOP, two arrangements of the Dukes, the Assassins, the final pair of Ambassadors, and the BOTTOM. This doesn't account for the Contessa block or two Captains, which is three items that must slotted into three slots (in the Duke block, between the Dukes and the Assassins, and between the Assassins and the Ambassadors) in some manner, but the Captains cannot be next to one another. In each slot, there is exactly one way to put all three items in the slot, which is $3 \cdot 1 = 3$ possible arrangements. In each slot, there are exactly two ways to put a Captain and the Contessa block in it and two ways to place the remaining captain in another slot, for $2 \cdot 2 = 4$ arrangements. If all three go in different slots, there are 3 ways to pick which slot gets the Contessa block. Thus, there are $3 + 4 + 3 = 10$ ways to arrange the items in the slots. There are the 2 arrangements of the Dukes and 2 of the Contessa block, for an answer of $2 \cdot 2 \cdot 10 = 40$.

95. In how many points does the graph of $y = e^x - 2$ intersect $y = \cos(8x)$?

A graph of the cosine function will go from $y = 1$ to $y = -1$ with a period of $\frac{2\pi}{8} = \frac{\pi}{4} \sim .79$. A graph of the exponential function will have a hypotenuse of $y = -2$, pass through $(0, -1)$, and head upwards; it will exit the region of interest at $(\ln 3, 1) \sim (1.1, 1)$. A quick sketch will show that these two functions will intersect three times.

96. Simplify $\sin 2j \cot j - \sec j \cos 2j$ by writing it using the fewest trigonometric functions. Once you have done this, rewrite it by rewriting csc , sec , and cot in terms of sin , cos , and tan .

This can be rewritten as $2 \sin j \cos j \cdot \frac{\cos j}{\sin j} - \frac{1}{\cos j} \cdot (2\cos^2 j - 1) = 2\cos^2 j - 2\cos j + \frac{1}{\cos j}$, which is just three trigonometric functions.

2018 Team Scramble
Solutions

97. What is the area, in square meters, of a triangle with sides measuring $\sqrt{5}$ m, $\sqrt{20}$ m, and $\sqrt{41}$ m?

Consider drawing this triangle in the Cartesian Plane. $\sqrt{5} = \sqrt{1+4} = \sqrt{1^2+2^2}$, $\sqrt{20} = \sqrt{4+16} = \sqrt{2^2+4^2}$, $\sqrt{41} = \sqrt{16+25} = \sqrt{4^2+5^2}$, so the triangle could be drawn using lattice points such as $(0,0)$, $(1,2)$, and $(5,4)$. Drawing a rectangle around this triangle, you can determine the area to be $5\cdot 4 - 1\cdot 2 - \frac{1}{2}\cdot 5\cdot 4 - \frac{1}{2}\cdot 1\cdot 2 - \frac{1}{2}\cdot 2\cdot 4 = 20 - 2 - 10 - 1 - 4 = 3$.

98. If $h(k) = \frac{(2k-1)^3}{(2-k)^2}$, evaluate $\frac{dh}{dk}$ when $k = 1$.

The quotient rule gives $\frac{dh}{dk} = \frac{(2-k)^2\cdot 3(2k-1)^2\cdot 2-(2k-1)^3\cdot 2(2-k)(-1)}{(2-k)^4} = \frac{1^2\cdot 3\cdot 1^2\cdot 2-1^3\cdot 2\cdot 1(-1)}{1^4} = 6 + 2 = 8$.

99. What is the largest possible area of a rectangle inscribed in an ellipse with equation $4x^2 + y^2 = 4$?

Sketching a tall ellipse containing a rectangle, we can write the $A = 2x\cdot 2y = 4xy$. Solving the ellipse for y gives $y = \sqrt{4-4x^2} = 2\sqrt{1-x^2}$, so that $A = 4x\cdot 2\sqrt{1-x^2} = 8x\sqrt{1-x^2}$. Taking a derivative gives $0 = 8\sqrt{1-x^2} + 8x\cdot\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = 8\sqrt{1-x^2} - \frac{8x^2}{\sqrt{1-x^2}}$. This becomes $1 - x^2 = x^2$, then $1 = 2x^2$ and $\frac{1}{2} = x^2$, so that $x = \frac{\sqrt{2}}{2}$. This makes $y = 2\sqrt{1-\frac{1}{2}} = \sqrt{2}$, for an answer of $A = 4\cdot\frac{\sqrt{2}}{2}\sqrt{2} = 4$.

100. Evaluate: $\int_2^5 h\sqrt{h-1}\,dh$

Let $u = h - 1$, so that $h = u + 1$ and $dh = du$. These let us rewrite the problem as $\int_1^4 (u+1)\sqrt{u}\,du = \int_1^4 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du = \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\Big|_1^4 = \frac{2}{5}(32-1) + \frac{2}{3}(8-1) = \frac{62}{5} + \frac{14}{3} = \frac{256}{15}$.