2017 Team Scramble
Solutions

Easier Problems

1. **What is the remainder when 2409 is divided by 16?**

   The standard algorithm gives 9.

2. **Evaluate: $\mathbf{8.67 \times 9.1}$**

   The standard algorithm gives 78.897.

3. **You have five favorite books, eight favorite stuffed animals, and four favorite beverages, but your dad says you can only bring one book and one stuffed animal, and that you must fill each of two differently-sized water bottles with a different beverage. How many different combinations of favorites can you bring on the trip?**

   Because of the different water bottle sizes, there are $4p2 = 12$ ways to choose the beverages, for an answer of $12 \cdot 5 \cdot 8 = 60 \cdot 8 = 480$.

4. **Express $\mathbf{.\overline{84}}$ as a reduced fraction.**

   If $x = .\overline{84}$, then $100x = 84.\overline{84}$, so that $99x = 84$ and $x = \frac{84}{99} = \frac{28}{33}$.

5. **If it is currently 4:46:57 PM, what time will it be in 1000 minutes? Answer in HH:MM:SS format including AM or PM.**

   1000 minutes is $\frac{1000}{60} = \frac{100}{6} = 16\frac{2}{3}$ hours, which is 16 hours and 40 minutes. In 16 hours, it'll be 8:46:57 AM, and 40 minutes later it'll be $9:26:57$ AM.

6. **What is the solution, in the form $(\boldsymbol{d, f, g})$, of the system of equations $\mathbf{2}\boldsymbol{d} + \boldsymbol{f} + \boldsymbol{g} = \mathbf{12}$, $\boldsymbol{d} + \mathbf{2}\boldsymbol{f} + \boldsymbol{g} = \mathbf{10}$, and $\boldsymbol{d} + \boldsymbol{f} + \mathbf{2}\boldsymbol{g} = \mathbf{6}$?**

   Adding the three equations gives $4d + 4f + 4g = 28$, so $d + f + g = 7$. Subtracting this from each equation gives $d = 5$, $f = 3$, and $g = -1$, for an answer of $(5,3,-1)$.

7. **Isaiah can build a cabinet in 8 hours, and Jessica can do it in 6 hours. To the nearest minute, how long would it take them to build the cabinet if they work together?**

   I's speed is $\frac{1}{8}$ cabinet per hour, and J's is $\frac{1}{6}$, so their combined speed is $\frac{1}{8} + \frac{1}{6} = \frac{3+4}{24} = \frac{7}{24}$, and it would take them $\frac{1}{\frac{7}{24}} = \frac{24}{7}$ hours working together, which is $\frac{24}{7} \cdot 60 = \frac{1440}{7}$ minutes, which is approximately 206.

8. **On average, nine chickens take three days to lay 99 eggs. To the nearest day, how many days would we expect it to take for three chickens to lay 79 eggs?**

   3 chickens can lay 33 eggs in 3 days, so 3 chickens can lay 11 eggs in 1 day, so 3 chickens could lay 77 eggs in 7 days, which is close enough for an answer of 7.

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9. Harry sees the Snitch 90 m away, and the chase is immediately on! If the Snitch flies directly away from Harry at 22 meters per second (mps) and Harry flies after it at 28 mps, how many seconds will it take Harry to catch the Snitch?

Harry must reduce a 90-meter gap at a net speed of $28 - 22 = 6$ meters per second, which will take $\frac{90}{6} = 15$ seconds.

10. What is the equation of the axis of symmetry of the parabola $\boldsymbol{y = -4(x+3)^2 - 1}$?

In this form, the axis of symmetry is whatever makes the squared term zero, which is $x = -3$.

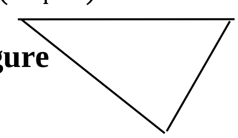
11. What are the coordinates, in the form (x, y) of the vertex of the parabola $\boldsymbol{y = -5x^2 - 9x}$?

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{-9}{2(-5)} = -\frac{9}{10}$, and the vertex will be at $y = -5\left(-\frac{9}{10}\right)^2 - 9\left(-\frac{9}{10}\right) = -\frac{405}{100} + \frac{81}{10} = \frac{405}{100} = \frac{81}{20}$, for an answer of $\left(-\frac{9}{10}, \frac{81}{20}\right)$.

12. What are the coordinates, in the form (x, y), of the rightmost x-intercept of $\boldsymbol{y = 48x^2 + 60x + 12}$?

We can write $0 = 48x^2 + 60x + 12$, then reduce it to $0 = 4x^2 + 5x + 1$, then factor it to $0 = (4x+1)(x+1)$ with roots at $x = -1$ and $x = -\frac{1}{4}$, for an answer of $\left(-\frac{1}{4}, 0\right)$.

13. What is the most specific geometric description you can give for the figure to the right?



It is an acute scalene triangle.

14. Consider a quadrilateral with three sides measuring 83 m, 25 m, and 8 m. If the fourth side of the quadrilateral is a whole number of meters, what is the product of its longest possible length and its shortest possible length?

Its shortest possible length is $83 - 25 - 8 + 1 = 51$, and its longest possible length is $83 + 25 + 8 - 1 = 115$, for an answer of $115 \cdot 51 = 5865$.

15. Consider a right circular cone with a base radius of 3 m and a height of 8 m. Now imagine that it is sliced by two vertical planes through the vertex so that the base of two of the resulting pieces is now a sector of the original circle with a central angle of 60°. What is the volume, in cubic meters, of one of these two pieces?

The volume should be $\frac{60}{360} = \frac{1}{6}$ of the entire cone, which is $\frac{1}{6} \cdot \frac{1}{3}\pi \cdot 3^2 \cdot 8 = 4\pi$.

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16. A quadrilateral with sides measuring 1 m, 2 m, 3 m, and 4 m is similar to a quadrilateral with sides measuring 2 m, 4 m, 6 m, and 8 m. If the area of the smaller quadrilateral is 3 m^2, what is the area, in meters, of the larger quadrilateral?

Because each dimension of the larger shape is 2 times that of the smaller, the larger area will be $2^2 = 4$ times that of the smaller, for an area of $4 \cdot 3 = 12$.

17. A triangle has sides measuring 25 m, 68 m, and 80 m. The angle bisector of the smallest angle is drawn to the opposite side, dividing it into two segments. What is the length, in meters, of the shorter of the two segments?

The smallest angle is opposite the smallest side, the 25. The angle bisector will divide it proportionally to the adjacent sides, so the shorter subsegment will be $\frac{68}{68+80} \cdot 25 = \frac{34}{74} \cdot 25 = \frac{17}{37} \cdot 25 = \frac{425}{37}$.

18. A folding screen with two panels that are each four feet wide is used to create a secluded space in the corner of a rectangular room. What is the maximum possible area, in square feet, of the floor of this space?

If we make the screen a flat 8-foot section, we could seclude a right triangle for a lot of space. However, if we pull the center fold out a bit, the ends will need to scoot in a smaller amount, and we can gain a bit more area. What will this shape be, particularly if I do it just enough to maximize it? Pretend that three other people are doing the same thing on the other sides of the corner, so that our folding screens collectively make an eight-sided shape. If all of us maximize our areas, we'll maximize the area of that octagon, so our answer will be one-fourth the area of a regular octagon with sides measuring 4, which will be $\frac{1}{4}\left(4^2 + 4 \cdot 4 \cdot \frac{4}{\sqrt{2}} + 4 \cdot \frac{4}{\sqrt{2}} \cdot \frac{4}{\sqrt{2}}\right) = \frac{1}{4}\left(16 + \frac{64}{\sqrt{2}} + \frac{64}{4}\right) = \frac{1}{4}\left(32 + 32\sqrt{2}\right) = 8 + 8\sqrt{2}$.

19. What is the name of the conic section with equation $\frac{(x+1)^2}{3} - \frac{(y-4)^2}{5} = 1$?

This is a hyperbola, due to the subtraction.

20. Evaluate: $\log_{32} \frac{1}{128}$

$$\log_{32} \frac{1}{128} = \log_{(2^5)}(2^{-7}) = -\frac{7}{5}\log_2 2 = -\frac{7}{5}$$

21. How many distinguishable functions are there that map the domain {-4, 4, -7} to the range {-6, -4, 7,0}?

Each of the three elements in the domain can be mapped to any of the four elements in the range, for an answer of $4 \cdot 4 \cdot 4 = 4^3 = 64$.

22. If $\log_5 2 = a$ and $\log_3 5 = b$, express $\log 3$ in terms of a and b?

$$\log 3 = \log_{10} 3 = \frac{1}{\log_3 10} = \frac{1}{\log_3 2 + \log_3 5} = \frac{1}{(\log_5 2)(\log_3 5) + b} = \frac{1}{ab + b}$$

23. Simplify: $\frac{14t^3 - 11t^2 + 33t - 54}{7t - 9}$

Factoring, $14t^3 - 11t^2 + 33t - 54 = (7t - 9)(2t^2 + t + 6)$, for an answer of $2t^2 + t + 6$.

24. Express the hexadecimal number $E9A$ in base four.

Each digit in base 16 is two digits in base 4, so $E = 14$ would be 32_4, 9 would be 21_4, and $A = 10$ would be 22_4, for an answer of 322122_4.

25. List all of the following numbers that are divisible by six:
7150, 82, 778, 8438, 517, 24, 189, 9879, 369, 40, 280, 3738

To be divisible by 6, you must be divisible by both 2 and 3. To be divisible by 2, you must end it an even number. To be divisible by 3, your digits must sum to a multiple of 3. By these rules, only 24 and 3738 are part of the answer.

26. What is the 37th term of an arithmetic sequence with a first term of 234 and a common difference of -96?

The 37th term is 36 differences away from the first term, for an answer of $234 + 36(-96) = 234 - 3600 + 144 = 378 - 3600 = -3222$.

27. What is the 5th term of a harmonic sequence with first term 8 and second term 6?

A harmonic sequence is some number divided by the terms of an arithmetic sequence. If we choose $8 \cdot 6 = 48$ as our number, the first two terms result from division by 6 and 8, after which we'd do 10, 12, and then 14, so our answer will be $\frac{48}{14} = \frac{24}{7}$.

28. What is the missing term of the sequence 1, 1, 4, 8, 37, ____, …?

There's no great way to tackle unusual sequences, you just have to fiddle around to see if you can find something "simple". In this case, we hoped the 1,1 would point you in a Fibonacci direction. Perhaps the rapid growth would then make you consider multiplying terms instead of adding them. $1 \times 1 = 1$, $1 \times 4 = 4$, and $4 \times 8 = 32$. The 32 is close enough to the 37 that it could be on the right track. It needs another 5, so perhaps the function is multiply and add 5 (which would make a difference of 0)? No, that would make the previous term 9 instead of 8 (a difference of 1), and the term before that 6 instead of 4 (a difference of 2). Hey, the differences are getting bigger with each term… instead of adding 5, perhaps we should be adding n? $1 \cdot 1 + 3 = 4$, $1 \cdot 4 + 4 = 8$, and $4 \cdot 8 + 5 = 37$, so the answer is probably $8 \cdot 37 + 6 = 296 + 6 = 302$.

29. What is the sum of the perfect squares less than 100?

This means the first 9 positive perfect squares, so we can use the equation $\frac{9(9+1)(2\cdot 9+1)}{6} = 3 \cdot 5 \cdot 19 = 15 \cdot 19 = 285$.

30. When 8 fair coins are flipped, what is the probability that exactly six of them show heads?

There are $2^8 = 256$ ways to flip 8 coins, and $8c6 = 8c2 = 28$ ways to arrange six heads among them, for a probability of $\frac{28}{256} = \frac{7}{64}$.

31. What is the equation of the line through the points $(\mathbf{1}, \mathbf{2}, \mathbf{3})$ and $(\mathbf{2}, \mathbf{4}, \mathbf{6})$? Express your answer in the form $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{y}) = \boldsymbol{h}(\boldsymbol{z})$.

The line goes in the direction $< 2-1, 4-2, 6-3 > = < 1, 2, 3 >$, which together with $(1,2,3)$ shows that the line goes through $(0,0,0)$, which makes the equation of the line $\frac{x-0}{1} = \frac{y-0}{2} = \frac{z-0}{3}$, which is $x = \frac{y}{2} = \frac{z}{3}$, which matches the form desired.

32. What is the median of the data set {5, 0, 3, 6, 7,}?

The median is the middle term when you arrange them in order, so it will be 5.

33. Set $\boldsymbol{F}$ is the set of all positive multiples of 4 less than 1000, and Set $\boldsymbol{G}$ is the set of multiples of 7 greater than 100. How many elements are in the set $\boldsymbol{F} \cap \boldsymbol{G}'$?

We want multiples of 4 less than 1000 that are not also multiples of 7 greater than 100, so we get all the multiples of 4 (249) minus the multiples of 28 from 100 to 1000. There are 35 less than 1000 and 3 less than 100, for an answer of $249 - 35 + 3 = 217$.

34. In the cross-math puzzle on the right, A-D are distinct digits (1-9) satisfying the four equations (two across, two down). What is the product of A, B, C, and D?

A	x	B	=	12
x		+		
C	+	D	=	15
=		=		
18		13		

A&B can be 2&6 or 3&4, and A&C can be 2&9 or 3&6, so that A is 2, 3, or 6, B is 6, 4, or 2, and C is 9, 6, or 3. If $C + D = 15$, the third case is not possible, and now D can be 6 or 9. Looking at $B + D = 13$, only the second case works, so that A, B, C, and D are 3, 4, 6, and 9, for a product of $3 \cdot 4 \cdot 6 \cdot 9 = 648$.

35. Evaluate: $\csc \frac{4\pi}{3}$

$\csc \frac{4\pi}{3} = \frac{1}{\sin \frac{4\pi}{3}} = \frac{1}{-\frac{\sqrt{3}}{2}} = \frac{2}{-\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

36. What is the area, in square meters, of a triangle with sides measuring 7 m, 4 m, and 6 m?

Heron's formula gives $\sqrt{\frac{17}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{9}{2}} = \frac{3}{4}\sqrt{17\cdot3\cdot5} = \frac{3\sqrt{255}}{4}$.

37. Express the complex number $(-8,8)$ in $re^{i\theta}$ form, where $0 \le \theta < 2\pi$.

$r = \sqrt{8^2+8^2} = 8\sqrt{1+1} = 8\sqrt{2}$, and the angle is in the second quadrant at $135° = \frac{3\pi}{4}$, for an answer of $8\sqrt{2}e^{\frac{3\pi i}{4}}$.

38. What is the value of y at the local maximum of the function $y(x) = 2x^3 + 10x^2 - 16x + 1$?

This cubic will go up to the right, down to the left, and wiggle in the middle, so its local maximum will be the leftmost place where the derivative is zero. $y'(x) = 0 = 6x^2 + 20x - 16$, reducing gives $0 = 3x^2 + 10x - 8 = (3x-2)(x+4)$, the leftmost root of which is $x = -4$ and gives $y = 2(-4)^3 + 10(-4)^2 - 16(-4) + 1 = -128 + 160 + 64 + 1 = 97$.

39. What is the average value of the function $y(x) = 2x^3 - x + 4$ on the interval $[3,5]$?

The average value is the integral over the interval divided by the width of the interval,

$$\int_3^5 (2x^3 - x + 4)dx = \frac{1}{2}x^4 - \frac{1}{2}x^2 + 4x\Big|_3^5 = \frac{1}{2}(625-81) - \frac{1}{2}(25-9) + 4(5-3)$$

$= 272 - 8 + 8 = 272$. Dividing this by the width of two gives 136.

Harder Problems

40. Evaluate: $\frac{16!\cdot10!\cdot6!}{9!\cdot5!\cdot12!\cdot9!\cdot5!}$

$$\frac{16!\cdot10!\cdot6!}{9!\cdot5!\cdot12!\cdot9!\cdot5!} = \frac{16\cdot15\cdot14\cdot13\cdot10\cdot6}{9!\cdot5!} = \frac{16\cdot15\cdot14\cdot13\cdot10\cdot6}{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot5\cdot4\cdot3\cdot2}$$
$$= \frac{13}{9\cdot8\cdot3} = \frac{13}{216}$$

41. Simplify by rationalizing the denominator: $\frac{12}{\sqrt[3]{4}-\sqrt[3]{2}+1}$

The denominator is of the form $c^2 - c + 1$, which may make you think of $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$. In this case $a = \sqrt[3]{2}$ and $b = 1$, and we should multiply by $\frac{\sqrt[3]{2}+1}{\sqrt[3]{2}+1}$ to get $\frac{12\left(\sqrt[3]{2}+1\right)}{2+1} = 4\sqrt[3]{2} + 4$.

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42. Because you have a 25%-off coupon, you go to a new restaurant and buy two dinners for $27.85 each, four appetizers for $7.85 each, and a flight of five artisanal root beers for $12.90. You're aware of the 10% sales tax and the 20% automatic gratuity (tip), but are surprised when the bill is higher than you were expecting. You ask your server, who explains that while the sales tax is calculated based on the discounted subtotal, the gratuity is calculated based on the pre-discount subtotal. How much is your bill, in dollars rounded to the nearest hundredth (cent)?

The subtotal is $2 \cdot 27.85 + 4 \cdot 7.85 + 12.90 = 55.70 + 31.40 + 12.90 = 100.00$, which is awfully convenient. : -) The gratuity would thus be 20.00, and the discounted subtotal will be 75.00, so that the sales tax will be 7.50, for an answer of $75.00 + 20.00 + 7.50 = 102.50$.

43. What value(s) of g satisfy $7g^2 + 8g - 12 = 0$?

Factoring gives $(7g - 6)(g + 2) = 0$, for an answer of $\frac{6}{7}$, -2.

44. Ron, Peter, and Siddhartha walk at constant speeds of 3 mph, 4 mph, and 5 mph, respectively. They decide to have an interesting "race" around a quarter-mile track. They all start at the same point facing in the same direction and begin walking. From then on, any time one of them ends up in the same place as another, the slower of the two must reverse direction. The "race" will continue until the first time that a specific pair is in the same place for the second time. I.e. if Ron & Peter meet up at position C, then later meet up at position B (which might be the same as C), the "race" is over. When the "race" ends, the winner is the person who is not in the same place as the other two. How long will the race take, to the nearest minute?

The fast one catches the slow one in $\frac{\frac{1}{4}}{2} = \frac{1}{8}$ hours, after which the middle one has traveled $4 \cdot \frac{1}{8} = \frac{1}{2}$ mile, so it at the starting line, while the others are exactly halfway around the track from there. Then the slow one turns around to meet the middle one in $\frac{\frac{1}{8}}{7} = \frac{1}{56}$ hours at a point $4 \cdot \frac{1}{56} = \frac{1}{14}$ miles "downstream" of the start, while the fast one is $\frac{1}{8} - 5 \cdot \frac{1}{56} = \frac{7}{56} - \frac{5}{56} = \frac{2}{56} = \frac{1}{28}$ miles "upstream". Thus, it will take another $\frac{\frac{3}{28}}{2} = \frac{3}{56}$ hours for the fast one to catch the slow one, who turned around again, for an answer of $\frac{1}{8} + \frac{1}{56} + \frac{3}{56} = \frac{11}{56}$ hours, which is $\frac{11}{56} \cdot 60 = \frac{11 \cdot 15}{14} = \frac{165}{14} = 11.7x \sim 12$ minutes.

45. The line through the points $(-4, -1)$ and $(6, 2)$ is perpendicular to the line through the points $(4, -2)$ and $(-6, v)$. What is the value of v?

The slope of the first line is $\frac{6-(-4)}{2-(-1)} = \frac{10}{3}$, so the slope of the second line must be $-\frac{3}{10} = \frac{4-(-6)}{-2-v}$. Cross-multiplying gives $6 + 3v = 100$, then $3v = 94$, for an answer of $v = \frac{94}{3}$.

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46. When the digits of a positive two-digit integer are reversed to form a new positive two-digit integer, the resulting number differs from one-third of the original number by less than one. What is the smallest possible value of the original number?

If the original number is TU worth $10T + U$, the new number is UT worth $10U + T$, so we can write $-1 < 10U + T - \frac{1}{3}(10T + U) < 1$, then $-1 < \frac{29U}{3} - \frac{7T}{3} < 1$, then $-3 < 29U - 7T < 3$. We'd like a small original number, so a small T, which implies a small U due to the structure of our inequality. If $U = 1$, T can be 4, for an original number of 41.

47. My piggy bank contains 46 coins worth a total of \$9.05. If the only coins that may be present are pennies, nickels, dimes, and quarters, what is the maximum number of dimes that could be in the piggy bank?

The average value of a coin is $\frac{905}{46} \sim \frac{900}{45} = 20$ cents, so we must have a lot of quarters, but the fewer quarters we have, the more dimes we can have. If we used only quarters and dimes, we could assume it was all quarters and work backwards to the number of dimes. 46 quarters would be worth $23 \cdot 50 = 1150$ cents, which is 245 more than we need. Every quarter we turn into a dime gets rid of 15 cents, so we could do it $\frac{245}{15} = \frac{49}{3} = 16\frac{1}{3}$ times, meaning perhaps we could have 16 dimes and 29 quarters, worth a total of $160 + 725 = 885$, leaving 20 cents, which is unfortunately not fixable by adding one smaller coin. We'll have to add a quarter, which means we now have five cents too many and thus can cut a dime and add a quarter, for an answer of 15.

48. A field contains humans, horses, and giant spiders! You see 89 heads, 294 legs, and 226 eyes (you think giant spiders have 8 legs, 8 eyes, and no head). How many giant spiders are there?

We can write $u + o = 89$, $2u + 4o + 8s = 294$, and $2u + 2o + 8s = 226$. The first combines nicely with the last, which we can reduce to $u + o + 4s = 113$, from which we subtract the first to get $4s = 113 - 89 = 24$, so that $s = 6$.

49. If (r, s, t, u) is a solution to the system of equations $-9r - 7s + 6t - 9u = -33$, $3r - 5s - 2t - 5u = -32$, and $-6r + 4s + 4t - u = 14$, what is the value of u?

Cancelling things that aren't u, let's start by tripling the second equation and adding it to the first to get $-22s - 24u = -129$. Then let's double it and add it to the third equation to get $-6s - 11u = -50$. Multiplying the first of these new equations by three and subtracting it from eleven times the latter gives $-121u - (-72u) = -550 - (-387)$, then $-49u = -163$, for an answer of $u = \frac{163}{49}$.

50. What is one solution, in the form (n, p, q), of the system of equations $n + p + q = 7$, $np + pq + nq = -42$, and $npq = -216$?

Focusing on the last equation, the numbers could be $\pm$1&1&216, 1&2&108, 1&3&72, 1&4&54, 1&6&36, 1&8&27, 1&9&24, 1&12&18, 2&2&54, 2&3&36, 2&4&27, 2&6&18, 2&9&12, 3&3&24, 3&4&18, 3&6&12, 3&8&9, 4&6&9, or 6&6&6. Of these possibilities, we could sum to seven with $18 - 12 + 1$ or $9 - 6 + 4$. Of these two, the latter satisfies $9(-6) + (-6)4 + 9 \cdot 4 = -54 - 24 + 36 = -78 + 36 = -42$, for an answer of $(9, -6, 4)$ or any permutation thereof.

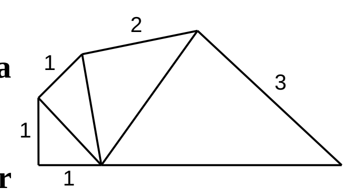
51. A cowboy must move his herd from his current camp at the origin, travel to the river (with equation $y = x + 2$) for water, and proceed to his next camp at $(2, -2)$. What is the minimum distance he can travel?

Even without the cool "reflection" trick, the alignment of this problem indicates that he should go straight to the river, which is a distance of $\frac{2}{\sqrt{2}} = \sqrt{2}$. After this, he should head straight to camp, which is $3\sqrt{2}$, for a total of $4\sqrt{2}$.

52. A gymnast launches himself from a springboard at a speed of 20 meters per second at an angle of $30°$ to the horizontal. How far away will he land, to the nearest meter, assuming the ground is horizontal?

Hopefully someone on the team knows that the acceleration due to gravity is approximately 9.8 m/s^2. The gymnast's initial velocity is $20\sin(30) = 20 \cdot \frac{1}{2} = 10$ upward and thus $10\sqrt{3}$ horizontally. It will take a bit more than one second for gravity to stop the gymnast's ascent, and the same for the gymnast to return to the ground, in which time he will have traveled $2 \cdot \frac{10}{9.8} \cdot 10\sqrt{3} = \frac{200 \cdot 1.732}{9.8} = \frac{346.4}{9.8} = 35.3x$ for an answer of 35.

53. Consider the figure to the right composed of right triangles. Each of the right triangles has a leg whose length in meters is a term from the Fibonacci Sequence, and whose other leg is the hypotenuse of the triangle whose leg is the previous Fibonacci term. Because the first triangle has no prior triangle to use for a leg, both of its legs are 1 meter. If this sequence is continued, what is the area, in square meters, of the sixth triangle?



One leg of the triangles follows the pattern 1, 1, 2, 3, 5, 8, so the sixth triangle has one leg of 8. The other leg is the hypotenuse of the previous triangle, and follows the pattern (1), $\sqrt{1^2 + 1^2} = \sqrt{2}$, $\sqrt{2 + 1^2} = \sqrt{3}$, $\sqrt{3 + 2^2} = \sqrt{7}$, $\sqrt{7 + 3^2} = \sqrt{16}$, $\sqrt{16 + 5^2} = \sqrt{41}$, for an answer of $A = \frac{1}{2} \cdot 8\sqrt{41} = 4\sqrt{41}$.

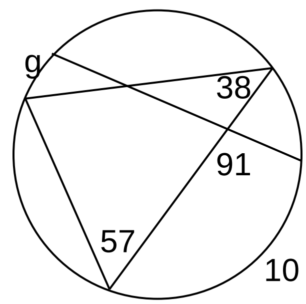
54. A plane is to be tessellated with a pattern that includes regular dodecagons. What is the minimum number of different types of regular polygons that must be used in such a pattern, including the dodecagon?

A dodecagon has 12 sides, so it has exterior angles of $\frac{360}{12} = 30°$. Drawing two of them side-to-side creates angles on each side equal to $2 \cdot 30 = 60°$, which is the interior angle of an equilateral triangle! We can alternate dodecagons and equilateral triangles around the perimeter of a dodecagon and create a tessellation with just two polygons.

55. The smallest possible circle is circumscribed about two tangent congruent circles with radii of 12 m. What is the side length, in meters, of the largest square that can be "inscribed" outside the two congruent circles but inside the circumscribed circle?

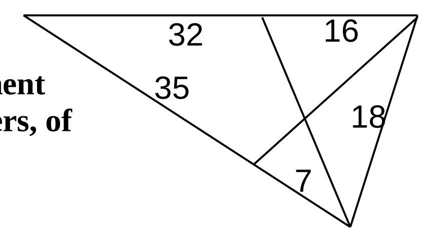
Drawing the three circles, a square could have one vertex pointing to the center of the figure, nestled between the two smaller circles. In this case, a 45-45-90 triangle with a hypotenuse ending on the centers of the two smaller circles will create a smaller square inside our square, and the sides of this smaller square can be shown to be $12\sqrt{2} - 12$. Drawing a radius of the large circle along the diagonal of these two squares, the diagonal of the large square can be shown to be $12 + \sqrt{2}(12\sqrt{2} - 12)$, so that a side will be $\frac{12}{\sqrt{2}} + 12\sqrt{2} - 12 = 18\sqrt{2} - 12$.

56. The figure to the right includes a circle and many chords, with some angle and arc measures given in degrees. What is the value of g?



The third angle of the large triangle will be $180 - 57 - 38 = 85$, so that the opposite arc will be $2 \cdot 85 = 170$, making the smaller sub-arc be $170 - 102 = 68$. The arc between this and g can be determined based on the 91° angle. $x + 102 = 2 \cdot 91 = 182$, so $x = 80$. The arc opposite the 38 will be $2 \cdot 38 = 76$, which means that $g = 360 - 76 - 170 - 80 = 110 - 76 = 34$.

57. The figure to the right shows a triangle with two cevians drawn through an interior point, with all perimeter segment lengths given in meters. What is the area, in square meters, of the smallest region created by these cevians?

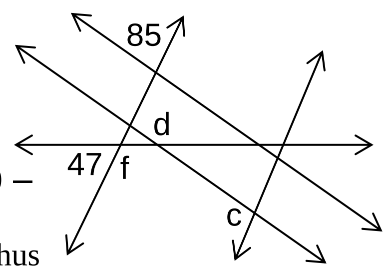


The 7 appears to be part of the smallest region, but we should make sure. The triangle with 7 and 18 will be $\frac{7}{35+7} = \frac{1}{6}$ of the large triangle, while the triangle with 16 and 18 will be $\frac{16}{32+16} = \frac{1}{3}$ of the large triangle, so the 16 will not be the smallest region. It's still possible that the 18 triangle is smaller than the 7 triangle at this point.

The large triangle is 42-48-18, which is 6 times a 7-8-3 triangle. The area will be $6^2 = 36$ times $\sqrt{9 \cdot 1 \cdot 2 \cdot 6} = 3 \cdot 2\sqrt{3} = 6\sqrt{3}$, which is $216\sqrt{3}$.

Drawing the third cevian through the point of intersection divides the 18 into two segments. Ceva's theorem allows us to write $\frac{7}{35} \cdot \frac{32}{16} \cdot \frac{a}{b} = 1$, which becomes $\frac{1}{5} \cdot \frac{2}{1} \cdot \frac{a}{b} = 1$, so that $\frac{a}{b} = \frac{5}{2}$, meaning that $a = \frac{5 \cdot 18}{7} = \frac{90}{7}$ and $b = \frac{2 \cdot 18}{7} = \frac{36}{7}$. We've now divided the large triangle into six smaller triangles. Labeling them A-F next to the 7, $\frac{36}{7}$, $\frac{90}{7}$, 16, 32, and 35, we can immediately write $F = 6A$, $E = 2D$, and $C = \frac{5}{2}B$. $A + B + C = A + \frac{7}{2}B = \frac{1}{6} \cdot 216\sqrt{3} = 36\sqrt{3}$, and $F + A + B = 6A + B = \frac{2}{7} \cdot 216\sqrt{3} = \frac{432\sqrt{3}}{7}$. Multiplying the second of these by $\frac{7}{2}$ gives $21A + \frac{7}{2}B = 216\sqrt{3}$, and subtracting the first gives $20A = 180\sqrt{3}$, so that $A = 9\sqrt{3}$.

58. The figure to the right shows five lines, two pairs of which are parallel. If all angle measures are given in degrees, what is the value of $f + d + c$?



We can start with $f = 180 - 47 = 133$, then see that $c = 180 - 85 = 95$, and finally deduce that $d = 47 + 85 = 132$ (as an exterior angle of the small triangle to the left). The answer is thus $133 + 132 + 95 = 265 + 95 = 360$.

59. A circular rug fits exactly into a 6-m square room. I'd like to buy a smaller circular rug that will fit exactly into a corner of the room, touching two walls and the large rug. What radius, in meters, should I instruct the rug-maker to make the smaller rug?

Looking at a diagonal of the square, which is a hypotenuse of a 45-45-90 triangle, we can say that $3 + r + r\sqrt{2} = 3\sqrt{2}$, then $r(1 + \sqrt{2}) = 3(\sqrt{2} - 1)$, then $r = \frac{3(\sqrt{2}-1)}{1+\sqrt{2}} = \frac{3(3-2\sqrt{2})}{1} = 9 - 6\sqrt{2}$.

60. A torus has an outer radius of 71 m and an inner radius of 35 m. What is the length in meters of the longest line segment that does not extend outside the torus?

In 2D, this is the standard concentric circles problem, giving an answer of $2\sqrt{71^2 - 35^2} = 2\sqrt{5041 - 1225} = 2\sqrt{3816} = 2 \cdot 2\sqrt{954} = 4 \cdot 3\sqrt{106} = 12\sqrt{106}$. The vertical dimension turns out not to matter due to the relative sizes of the chord and the curvature of the passage.

2017 Team Scramble
Solutions

61. What is the smallest number of sides a regular polygon can have if the number of diagonals that can be drawn in it is greater than the measure, in degrees, of one of its interior angles?

We can write $\frac{n(n-3)}{2} > 180 - \frac{360}{n}$, multiply to get $n^3 - 3n^2 > 360n - 720$, then $n^3 - 3n^2 - 360n + 720 > 0$. This may factor, but there's no guarantee, so I'm going to start guessing answers… $n = 10$ will be too small, as the middle terms are both subtracting. $n = 20$ gives $8000 - 1200 - 7200 + 720$, which barely works, so we're close. $n = 19$ gives $361 \cdot 19 - 3 \cdot 361 - (7200 - 360) + 720 = 6496 - 6840$, which doesn't work, so the answer is 20.

62. What is the largest number of regions into which a plane can be divided by a quadrilateral, an ellipse, and two lines?

Consider the quadrilateral and ellipse first. Using a convex quadrilateral like a parallelogram, we can create 10 regions. Using a concave quadrilateral like a dart, we can also create 10 regions, but these ten regions are "lined up" better… We can draw our first line straight through the outside and five regions inside the ellipse to create six new regions for a subtotal of 16, then another that intersects that to create eight new regions, for an answer of 24.

63. A cube of blue plastic is painted green, then cut into 38 smaller cubes. How many of these smaller cubes have exactly two green sides?

Obviously, something weird is going on with this problem, as 38 isn't a perfect cube. The catch is that the smaller cubes are not congruent. To be able to form a larger cube, however, they must have a relationship to one another, where their dimensions are rational multiples of one another. If we divided a cube congruently, we could get 8, 27, 64, 125, etc. smaller cubes. The 8 and 27 are too small for this analysis, but the 64 might work… might some of those 64 cubes combine to form a larger cube but smaller than the original? If so, we might make up to eight 2x2x2 cubes out of 8 cubes at a time, or up to one 3x3x3 cube out of 27 cubes. In the latter case, there would be $64 - 27 =$37 cubes left over, which would make $37 + 1 = 38$ cubes! Thus, we've divided a 4x4x4 cube into one 3x3x3 cube and 37 1x1x1 cubes. The 3x3x3 will have three painted sides, as will seven 1x1x1s. The cubes with two sides will be on $12 - 3 = 9$ edges, two along each edge, for a total of $2 \cdot 9 = 18$.

64. What is the length of the latus rectum of $y = 4x^2 - 3x + 2$?

The latus rectum is the "width" of a conic section perpendicular to its major axis through its focus. The axis of the parabola is $x = -\frac{b}{2a} = -\frac{-3}{2\cdot 4} = \frac{3}{8}$, so the vertex at $y = 4\left(\frac{3}{8}\right)^2 - 3\left(\frac{3}{8}\right) + 2 = \frac{4\cdot 9}{64} - \frac{9}{8} + 2 = -\frac{9}{16} + \frac{32}{16} = \frac{23}{16}$. The focal distance $p = \frac{1}{4a} = \frac{1}{4\cdot 4} = \frac{1}{16}$ and the length of the latus rectum is equal to $4p = 4 \cdot \frac{1}{16} = \frac{4}{16} = \frac{1}{4}$.

65. Chris measures 589.6 kg of Dixium at 12:01 AM on January 1st. If Dixium has a half-life of 137 minutes, what is the first day (month & day) on which Chris will have less than 1 μg of Dixium?

In grams, we can write $589600\left(\frac{1}{2}\right)^n < \frac{1}{10^6}$, where n is the number of half-lives to pass. This becomes $\left(\frac{1}{2}\right)^n < \frac{1}{5.896\times10^{11}}$, which we can flip to say $2^n > 5.896\times10^{11}$, so that $n = \log_2 5.896\times10^{11} = \log_2 5.896 + \log_2 10^{11} = \log_2 5.896 + 11\log_2 10 = \log_2 5.896 + 11(\log_2 2 + \log_2 5)$. This can be approximated as $2.6 + 11(1+2.4) = 2.6 + 11\cdot3.4 = 2.6 + 37.4 = 40$ half-lives, which is $\frac{40\cdot137}{60} = \frac{2}{3}\cdot137 = \frac{274}{3} = 91\frac{1}{3}$ hours, which is five hours short of four days, for an answer of January 4th.

66. When one cow is corralled in a 1 m square corral with 2-cm tall grass, it takes it one day to eat the grass down to the ground. When two cows are corralled in a 2-m square corral with 1-cm tall grass, it takes them two days to eat the grass down to the ground. How many days would it take four cows corralled in a 3-m square corral with 5-cm tall grass to eat the grass down to the ground?

The first two pieces of information appear to give contradictory rates at which the cows eat, which hopefully caused people to consider that perhaps the grass was growing. For the one cow case, we can write that the total amount of grass eaten by the cow, $1\cdot1\cdot c$, where c is the amount of grass a cow eats in a day, is equal to the total amount of grass to have been in the pasture by the end, $1^2(2+1g)$, getting $c = 2+g$. For the second case, we can write $2\cdot2\cdot c = 2^2(1+2g)$ to get $4c = 4+8g$, which is equivalent to $c = 1+2g$. Substituting for c gives $2+g = 1+2g$, then $1 = g$, so that $c = 3$. Now we can figure out the third case. $4\cdot d\cdot3 = 3^2(5+d\cdot1)$, which becomes $12d = 45+9d$, so that $3d = 45$, giving $d = 15$.

67. What is the largest real value of v satisfying $3v^2 + 3vw + 3w^2 = 4$?

If v gets too large, the discriminant for w will become negative, meaning there are no solutions for those values of v. Thus, we want the (larger) value of v that makes w's discriminant zero. Rewriting as a quadratic in terms of w gives $3w^2 + (3v)w + (3v^2-4) = 0$, the discriminant of which will be $(3v)^2 - 4\cdot3(3v^2-4) = 9v^2 - 36v^2 + 48 = -27v^2 + 48 = 0$, so that $v^2 = \frac{48}{27} = \frac{16}{9}$, so that $v = \pm\frac{4}{3}$, for an answer of $\frac{4}{3}$.

68. What is the largest possible area of a triangle with vertices on the parabola $y = -x^2 + 16x - 48$ and with non-negative x & y coordinates?

The parabola factors to $y = -(x^2 - 16x + 48) = -(x-12)(x-4)$ with roots at 12 and 4 and thus an axis of symmetry of $x = \frac{12+4}{2} = 8$ and a vertex height of $y = -8^2 + 16\cdot8 - 48 = -64 + 128 - 48 = 16$. The largest triangle will use those points and thus have a base of $12-4 = 8$ and a height of 16, for an area of $A = \frac{1}{2}bh = \frac{1}{2}\cdot8\cdot16 = 64$.

69. Express the product of the base nine numbers 486_9 and 687_9 in base nine.

You can multiply within a base with the same algorithm as in base ten, so long as your "carrying" or "regrouping" is in that base. For example, $7_9 \times 6_9 = 42$, which is 46_9, so we'd write a 6 and carry a 4. In the end, the answer is 375,506.

70. What is the smallest number greater than 1000 that is divisible by 9 but leaves a remainder of 8 when divided by 11?

We're looking for multiples of 9 that are three less than a multiple of 11. Focusing on the larger number, we'll consider 8, 19, 30, 41, 52, 63, aha! 63 is a multiple of 9 and three less than a multiple of 11. Now we can go up by the LCM of 9 and 11, which is 99, until we're over 1000. $63 + 990 = 1053$, which is our answer.

71. How many positive integers are factors of 14040?

The prime factorization is $14040 = 2^3 \cdot 1755 = 2^3 \cdot 5 \cdot 351 = 2^3 \cdot 3^2 \cdot 5^1 \cdot 39 = 2^3 \cdot 3^3 \cdot 5^1 \cdot 13^1$, so it has $4 \cdot 4 \cdot 2 \cdot 2 = 64$ factors.

72. How many positive integers are factors of both 5268 and 18438?

The prime factorizations are $5268 = 2^2 \cdot 1317 = 2^2 \cdot 3^1 \cdot 439^1$ and $18438 = 2^1 \cdot 9219 = 2^1 \cdot 3^1 \cdot 7^1 \cdot 439^1$. Every number that is a factor of both 5268 and 18438 would need to be a factor of their GCF, which is $2^1 \cdot 3^1 \cdot 439^1$, which has $2 \cdot 2 \cdot 2 = 8$ factors.

73. John writes the number 1 and every 54th counting number after that (55, 109, etc.). Jane writes the number 47 and every 72nd counting number after that (121, 195, etc.). What is the smallest positive difference between a number on John's list and a number on Jane's list?

John's number can vary by any multiple of 54, and Jane's can vary by multiples of 72. Because the GCF of 54 and 72 is 18, the difference between John's number and Jane's number is always changing by a multiple of 18, no matter how many 54s or 72s are taking place. Their numbers start $47 - 1 = 46$ apart, but this can change by any multiple of 18, so they could become 28 or 10 apart, but 10 is not the answer, as they could also end up -8 apart, which represents John's number being larger, so the answer is 8.

74. My favorite number is a four-digit counting number with the interesting property that the last two digits form a two-digit counting number that is twice the two-digit counting number formed by the first two digits. In addition, the central two digits form a two-digit counting number that is three times the two-digit counting number formed by the first two digits. What is the largest number that could be my favorite number?

If the number is $ABCD$, then $CD = 2AB$ and $BC = 3AB$, so $B > C > A$, D is even, and $B + C$ is a multiple of 3. We're looking for the largest possible number, so we'd like a large A. Because we need to be able to triple AB, we could try $A = 3$, but because B must be greater than A it would have to be at least 4, which doesn't let AB triple to a two-digit number. So we can try $A = 2$, which could result in B being 6, 7, or 8, so we might end up with 26 which triples to 78 (no), 27 which triples to 81 (possible, but C should be bigger than A, so no), or 28 which triples to 84 (possible). If this third case holds, CD would need to be twice 28, which would be 56, so this third case doesn't hold. Alas, we must use $A = 1$, so that B is 2, 3, or 4, in which case 12 could triple to 36 (no), 13 could triple to 39 (possible, but C should be less than B, so no), or 14 which triples to 42 (possible). If this third case holds, CD would need to be twice 14, which would be 28, which works, making our answer 1428.

75. What is the tens digit of 678^{90}?

First off, this will be the same as for $78^{90} = (80 - 2)^{90} = (-2)^{90} + 90 \cdot 80^1 \cdot (-2)^{89} + \cdots$, so it'll have the same tens digit as $(-2)^{90} = 2^{90} = 1024^9 = 24^9 = 4^9 + 9 \cdot 20^1 \cdot 4^8 + \cdots$. $4^9 = \frac{1024^2}{4} = \frac{1048476}{4} = 262144$ and $9 \cdot 20 \cdot 4^8 = 180 \cdot 4^8$ will end in zero and have a tens digit based on 8 times the last digit of 4^8, which is 6, giving $8 \cdot 6 = 48$, whose last digit is 8, so the answer is $4 + 8 = 12$, meaning 2.

76. What is the sum of the first 99 terms of an arithmetic sequence with first term -53 and common difference 25?

The last term is $-53 + 98 \cdot 25 = -53 + 2500 - 50 = 2500 - 103 = 2397$, for an answer of $\frac{99}{2} \cdot 2344 = 99 \cdot 1172 = 117200 - 1172 = 116{,}028$.

77. What is the sum of the natural numbers from 78 to 516 inclusive?

There are $516 - 78 + 1 = 439$ numbers, with "outer pairs" adding to $78 + 516 = 594$, for a total of $\frac{439}{2} \cdot 594 = 439 \cdot 297 = (300 - 3)439 = 130{,}383$.

78. Sequence U is an arithmetic sequence with first term 64 and common difference 28. Sequence T is a geometric sequence with first term 9 and common ratio 4. How many numbers less than 1000 are common both sequences?

Focus on T, as there will be fewer to consider. 9, 36, 144, 576, that's it. Subtract 64 from the latter two to get 80 and 512, neither of which is a multiple of 28, giving an answer of 0.

79. Bag S contains 4 purple marbles and 3 blue marbles. Bag R contains 4 purple marbles and 1 blue marble. Two marbles are drawn randomly from Bag S and placed in Bag R. What is the probability that a marble then drawn from Bag R is blue?

We might transfer PP, PB, or BB from S to R, with probabilities of $\frac{6}{21} = \frac{2}{7}$, $\frac{12}{21} = \frac{4}{7}$, and $\frac{3}{21} = \frac{1}{7}$. After these transfers, the probabilities of getting a blue are $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$, for an answer of $\frac{2}{7} \cdot \frac{1}{7} + \frac{4}{7} \cdot \frac{2}{7} + \frac{1}{7} \cdot \frac{3}{7} = \frac{13}{49}$.

80. Your trusted friend James is dealt two cards from a standard 52-card deck. He confides that he does not have two face cards. What is the probability that he has a pair?

Under normal circumstances, there would be $52c2 = \frac{52 \cdot 51}{2} = 26 \cdot 51 = 1326$ possibilities for James' cards, but I believe he doesn't have two face cards, so there are $12c2 = 6 \cdot 11 = 66$ sets he doesn't have, for an actual total of 1260. There are 10 ranks he still might have a pair in, and in each rank there are $4c2 = 6$ ways to pick two of them, for an answer of $\frac{60}{1260} = \frac{6}{126} = \frac{1}{21}$.

81. Your keyring has six keys on it, each of which is one-sided (has a flat side and a bumpy side). If two of the keys are identical, how many distinguishable arrangements are possible?

There is 1 way to place the first unique key, 5 for the second, then 4, then 3, after which the two identical keys can only be placed one way, for a total of $1 \cdot 5 \cdot 4 \cdot 3 = 60$ arrangements. Arbitrarily call the orientation of one key "up". Every other key might be up or down, for an answer of $60 \cdot 2^5 = 60 \cdot 32 = 1920$. However, each of these arrangements is actually the duplicate of another, as the entire key ring can be turned over, so the real answer is 960.

82. At the Puzzlebrary, there is a shelf of decorative books. There are eight green books, five yellow books, three orange books, and one red book. Five of the green books, two of the yellow books, and one orange book do not have writing on their spines, so they could be placed on the shelf in two possible orientations. Of these blank-spined books, three of the green ones have had their front and back covers glued together to form one giant book, and the two yellow ones have been similarly joined. In how many ways could all of these books be placed next to one another on the shelf if books of the same color must be kept together?

There are $4! = 24$ ways to arrange the colors. There's only 1 normal red book, so only 1 way to arrange it in its color. There are three orange books, so there are $3! = 6$ ways to arrange them, but one of them has a blank spine and could be oriented two ways, making $2 \cdot 6 = 12$ ways to arrange the oranges. There are five yellow books, but because the two blank-spined books are glued together, there are really only four yellow books, one of which has two possible orientations, for a total of $2 \cdot 4! = 48$ ways to arrange yellows. Similarly, there are really only six green books, three of which are blank-spined, for a total of $2^3 \cdot 6! = 8 \cdot 720 = 5760$ green arrangements and an answer of $5760 \cdot 48 \cdot 12 \cdot 1 \cdot 24 = 79{,}626{,}240$.

2017 Team Scramble
Solutions

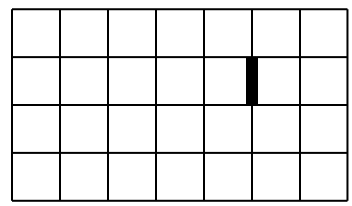
83. When some people are surveyed about cookie ingredients, 195 say they like chocolate chips, 175 say they like nuts, and 183 say they like coconut. As it turns out, 61 of them like both chocolate chips and coconut, 49 like both nuts and coconut, and 64 of them like both chocolate chips and nuts. If 9 of them like all three ingredients and 4 people like none of them, how many people were surveyed?

In a traditional 3D Venn diagram, the 9 and 4 are easy to enter. The 2D intersections are then 52, 40, and 55, after which coconut alone is 82 and nuts alone is 71, at which point the answer is $195 + 40 + 71 + 82 + 4 = 392$.

84. Sai and Talia play a game where they take turns drawing cards (without replacement) from a standard 52-card deck. If the person who draws the first 9 wins, what is the probability that the first player wins on their second turn (the third turn of the game)?

It must go not-9, not-9, 9, for an answer of $\frac{48}{52} \cdot \frac{47}{51} \cdot \frac{4}{50} = \frac{12}{13} \cdot \frac{47}{51} \cdot \frac{2}{25} = \frac{4}{13} \cdot \frac{47}{17} \cdot \frac{2}{25} = \frac{376}{5525}$.

85. The figure to the right shows an array of unit squares, with one edge thickened. How many paths from the upper left corner to the lower right corner along the gridlines pass through this edge, do not use any edge twice, and have a length of 13?



There are 11 steps (7 to the right, 4 down) between the two corners, so we're going two steps out of our way (once the wrong way, once to undo it) if we take 13 steps. Without the thickened edge, this means we'd be interested in arrangements of RRRRRRRDDDDUD or RRRRRRRDDDDRL. In this case, however, we might: take the misstep before reaching the top of the edge and proceed normally from the bottom of the edge, get to the top normally and take the misstep after the bottom, or the thickened edge might _be_ the misstep.

In the first case, we might go RRRRRDUD or RRRRDRL, then D the thickened edge, then RRDD. There are $4c2 = 6$ ways to do the last, one way to do the middle step, and we must think a bit about the first section. For the extra UD subcase, there must be at least one D before the U, and there must be at least one step after the U (or the U would be going up the thickened edge, which is a different case), and the U cannot appear immediately before or after a D (or the same segment would be traversed twice). Thus, the sequence must be something like ?D?RUR?, where the ?'s collectively contain RRRD in some order. The DUD must be in the order DDU or DUD, and the U must have Rs on both sides (DDRUR or DRURD), leaving three Rs to choose which of four slots to slip into (?D?D?RUR? or ?D?RUR?D?). The three Rs can group together or separate as they like, so there are $(3 + 3)c3 = 6c3 = 20$ ways for them to go in each case, giving $2 \cdot 20 = 40$ routes for the first subcase. Similarly, to get to the top with a RL misstep, we must have at least one R before the L, the L _could_ be last, and the L cannot appear immediately before or after an R. Because there's only one D, the L _must_ be last in this sequence (or it would be immediately followed by an R), so there is just one way to make this misstep before the edge. Thus, there are $60 + 1 = 41$ ways to reach the top of the edge, 1 way to go down it, and 6 ways to get to the destination, for a first-case subtotal of $41 \cdot 6 = 246$.

In the second case, there are $6c1 = 6$ ways to reach the top, 1 way to go down the thickened edge, and some case analysis to be done afterwards. For the extra UD subcase, we're looking at RRDDUD, but the U cannot be the first step, must be followed by at least one D, but cannot immediately precede or follow a D. This essentially means we must have DDRURD, DRURDD, or RURDDD, which is 3 cases. Similarly, for the extra RL subcase we're dealing with RRDDRL, but the L _can_ be the first step, must have at least one R after it, and cannot be right next to an R. Thus, we can start with LD, in which case there are $4c1 = 4$ ways to arrange the remaining Rs and D. If we don't start with LD, then we must contain DLD, which can be mixed with the Rs in 3 ways, because we must have at least one R after the L, for a total of $4 + 3 = 7$ options. Thus, there are $6(3 + 7) = 6 \cdot 10 = 60$ ways for the second case to work out.

In the third sub-case, the misstep is a U on the thickened edge. To get to the bottom of the edge required RRRRRDD, but the last step must be an R, leaving $6c2 = 15$ ways to get there. Similarly, from the top to the end must be RRDDD, but the first step must be an R, leaving $4c1 = 4$ ways to do the rest, for a third-case subtotal of $4 \cdot 15 = 60$, and an answer of $246 + 60 + 60 = 366$.

86. In one phase of a new board game, four tiles are placed face down. It is known that two of the tiles have 0 on their faces, while the other two have 1000 on their faces. You place a cube on top of a tile, then your opponent looks at the other three tiles, and must flip exactly one tile with a 0 on its face over so that you can see the 0. At this point, you can take the tile with your cube on it or either of the other two face-down tiles. If your goal is to get a tile with 1000 on its face and you choose wisely, what is the probability that you achieve your goal?

At the time I place my cube, there is a $\frac{1}{2}$ probability that I've picked a 1000 tile, which I could also think of as that tile being half of a 1000 tile and half of a 0 tile (kind of like Schrodinger's Cat : -). The other three tiles are essentially the same as this one at this point. Then my opponent looks at the other tiles and reveals a 0. She can always do this, so it doesn't tell me anything about my tile, but it _does_ tell me something about the other tiles. Between them, they had $\frac{3}{2}$ of a 1000 tile and $\frac{3}{2}$ of a 0 tile, and now I know where two of those three halves are. This means the remaining two tiles without cubes have $\frac{3}{2}$ of a 1000 tile and $\frac{1}{2}$ of a 0 tile between them, which is like saying each of them has a $\frac{3}{4}$ probability of being a 1000 tile. Because the tile with the cube on it only has a $\frac{1}{2}$ probability of being a 1000 tile, I should take either of the other tiles, for an answer of $\frac{3}{4}$.

87. A bubblegum machine has 21 red gumballs, 7 purple gumballs, 37 orange gumballs, 6 blue gumballs, 89 yellow gumballs, and 8 green gumballs. My four children all want gumballs, but two of them are not willing to chew the same color as anyone else. How many gumballs must I be willing to buy in order to be certain I that I can distribute a gumball to each of my children according to their demands?

I need to take enough gumballs that I can be sure I'll get at least three different colors. The worst thing that could happen is that I'd get 89 yellows, then 37 oranges, and finally something else, for an answer of $89 + 37 + 1 = 127$.

88. What is the equation of the plane through the points $(3, 1, 2)$, $(8, 2, 1)$, and $(1, 0, 2)$? Express your answer in the form $Ax + By + Cz = D$, where $A > 0$ and A, B, and C are collectively relatively prime integers.

The vector from the first point to the second is $< 5,1,-1 >$, and from the first to the third is $< -2,-1,0 >$. Their cross product is $< -1,2,-3 >$, which is perpendicular to both and thus perpendicular to the plane, as would be $< 1,-2,3 >$ which fits the desired form of the answer better. The equation of the plane is thus $x - 2y + 3z = d$. Substituting the third point gives $1 + 3 \cdot 2 = d = 7$, for an answer of $x - 2y + 3z = 7$.

89. What is the shortest distance between the lines $\frac{x-7}{3} = \frac{y}{2} = \frac{z-9}{4}$ and $x - 2 = \frac{y+1}{2} = \frac{z+6}{3}$?

The lines in vector form are $< 7,0,9 > + < 3,2,4 > t$ and $< 2,-1,-6 > + < 1,2,3 > s$ and go in the directions $< 3,2,4 >$ and $< 1,2,3 >$, so the shortest distance will happen in the direction perpendicular to both of these, found by doing the dot product: $< 2 \cdot 4 - 3 \cdot 2, 3 \cdot 3 - 1 \cdot 4, 1 \cdot 2 - 2 \cdot 3 > = < 2,5,-4 >$. The distance from the origin to the plane containing the first line and perpendicular to this vector is $\frac{<7,0,9> \cdot <2,5,-4>}{\sqrt{2^2+5^2+4^2}} = \frac{14-36}{\sqrt{45}} = -\frac{22}{3\sqrt{5}}$, and the same distance for the second line is $\frac{<2,-1,-6> \cdot <2,5,-4>}{\sqrt{2^2+5^2+4^2}} = \frac{4-5+24}{\sqrt{45}} = \frac{23}{3\sqrt{5}}$. The difference between these is $\frac{45}{3\sqrt{5}} = \frac{45\sqrt{5}}{15} = 3\sqrt{5}$.

90. In a data set of seven integer test scores from 0 to 100 inclusive, the unique mode is 90, the median is 55, and the range is 45. What is the smallest possible value of the mean?

Let the seven scored in ascending order be A, B, C, 55, E, F, and $A + 45$. Obviously, at least two of the higher three will be 90. To have a small mean, we'd like all the elements to be small, and for A to be small, $A + 45$ would also need to be small. This leads us to the set 45, 46, 47, 55, 56, 90, 90, with a mean of $\frac{138+111+180}{7} = \frac{429}{7}$.

91. In the set of integers {31, 6, 130, 93, 605, 99, x, y, z}, the median is less than the mode, which is less than the mean. What is the smallest possible value of $x + y + z$?

In order, the known elements are 6, 31, 93, 99, 130, and 605, and there are a total of 9 elements, so the median will be the fifth of them. That means the median might be anywhere from 31 to 130. Because the mode must be greater than the median, we can't actually put all of x, y, and z below 31; at least one needs to be above, so the median can actually only be from 93 to 130. Because we want a low total of x, y, and z, we also want a low mean, but it must be bigger than the mode, which is bigger than the median. This might work out if x & y are really low, and $z = 99$ (making the median 93 and the mode 99). The mean in this case would need to be greater than 99, so the total of the elements would need to be greater than $9 \cdot 99 = 891$, the lowest such total being 892. Even though we know $z = 99$, we'll express the actual total as $6 + 31 + 93 + 99 + 130 + 605 + (x + y + z) = 964 + (x + y + z) = 892$, so that $x + y + z = 892 - 964 = -72$. z can be 99, and x and y will be whatever low and/or negative numbers they like to create this sum.

92. The average grade in Professor Slughorn's Potions class is 45, although the average among Muggleborns is 51. If his class is 80% purebloods (thus 20% Muggleborns), what is the average grade among the purebloods?

The overall average will be the weighted average of the two groups, so we can write $45 = \frac{1}{5} \cdot 51 + \frac{4}{5}p$, which becomes $\frac{225-51}{5} = \frac{4p}{5}$, so that $174 = 4p$ and $p = \frac{174}{4} = \frac{87}{2}$.

93. In the cryptarithm below, each instance of a letter represents the same digit (0-9), and different letters represent different digits. E.g. if one A is a 1, all A's are 1's and B cannot be 1. What is the smallest possible value of the five-digit number ABCDE? Note: numbers cannot have zero as their leading digit.

$$\begin{array}{r} \boldsymbol{AB} \\ \underline{-\boldsymbol{BC}} \\ \boldsymbol{DE} \end{array}$$

We'd most like a low A, but A, B, and D need to be different digits, so we can try $A = 3$ with B & D being 1 & 2… Neither works, so we need to move up to $A = 4$, which should work. A low B is the next most-important thing, so we'll try 1, probably making $D = 2$. A low C is next most-important, so we'll try $C = 3$, which makes $E = 8$. The problem is thus $41 - 18 = 23$, for an answer of 41328.

94. A Crime has been committed, and the Usual Suspects are brought in for "questioning", in which they each make one statement.

A: I didn't do it!
B: C&D did it!
C: D didn't do it.
D: Neither A nor E did it.
E: I (Person E) did it.

Forensics has determined that two people worked together on the Crime (and that they're among A-E), and the polygrapher says that there were exactly two lies told, although he's forgotten which statements those were. List all suspects (by letter, A-E) that you can be certain are guilty of the Crime.

There are exactly two lies and thus three true statements. If A were lying, then A was one of the people who did it, so D is also lying, so everyone else must be telling the truth, which is not possible, as B is claiming C&D did it (not A). Thus, A is telling the truth (and isn't part of our answer), and the others are two lies and two truths.

If B is telling the truth, C and E must both be lying, so that D is the other truth-teller. This is consistent, so C and/or D can be part of our answer if they show up in all other consistent interpretations of the statements (those in which B is lying).

If B is lying, there are two truths and a lie left over, and at least one of C & D are innocent. If C were the other liar, D would be guilty and C innocent, and D & E would both have to be truth-tellers, which is impossible (they disagree on E's guilt), so if B is lying, C must be telling the truth. This would leave a truth and a lie for D&E, and we know that A&D are innocent, so two of BC&E are guilty. D could be telling the truth with E lying, in which case the guilty are B&C. D could be lying with E telling the truth, in which case the guilty are E and one of B&C.

Thus there are three consistent interpretations of the statements:
TTFTF means C&D are guilty.
TFTTF means B&C are guilty.
TFTFT means E and one of B&C are guilty.

Thus, there is no suspect you can be _sure_ is guilty.

95. In the land of Knights & Knaves, Knights always tell the truth and Knaves always lie. You meet a group of six residents who make the following statements:

A: There are exactly two Knights here.
B: There are at least two Knaves here.
C: At least one of A&B are Knights.
D: There is an even number of Knaves here.
E: There are at most four Knights here.
F: At least one of C & D are Knaves.

List all residents (by letter, A-F) that you can be certain are Knights.

If we assume that A is true, then A is one of two Knights. This makes B's statement true, too, so B is the other Knight; everyone else would have to be Knaves. But C's statement is also true, even though he should be a Knave, which is a contradiction meaning we were wrong when we assumed A was true, so A must be a Knave, and there ARE NOT exactly two Knights.

If we assume that B is false, then B is one of at most one Knaves, meaning everyone else would have to be Knights, but we've already proved that A is a Knave, so this is a contradiction and B is a Knight, and there ARE at least two Knaves. This makes E's statement true, so E must be a Knight. Also, because B is a Knight, C is telling the truth and is thus also a Knight. Because C is a Knight, and because of F's statement about C & D, D & F must be one Knight and one Knave. Thus, there are a total of four Knights and two Knaves, which means that D is a Knight and F is a Knave, so that our answer is BCDE.

96. Evaluate in radians: Arctan $\left(-\frac{\sqrt{3}}{3}\right)$

You can think of this as an angle in a triangle with an opposite side of $\pm\sqrt{3}$ and an adjacent side of ∓ 3, with a hypotenuse of $\sqrt{3^2+\left(\sqrt{3}\right)^2}=\sqrt{9+3}=\sqrt{12}=2\sqrt{3}$. Dividing by $\sqrt{3}$, the sides are 1, $\sqrt{3}$, and 2, which is a 30-60-90 triangle with our angle being the $30°=\frac{\pi}{6}$. Now we need to consider the range of the Arctan function, which is $-\frac{\pi}{2}<\text{Arctan}(x)<\frac{\pi}{2}$, and the fact that our $\pm$ triangles were in quadrants II or IV, for an answer of $-\frac{\pi}{6}$.

97. A five-meter ladder is on level ground leaning against a vertical wall, when suddenly its feet begin to slide away from the wall at a constant speed of 5 meters per second. At the moment the feet are 4 meters from the wall, how fast is the top of the ladder sliding down the wall, in meters per second?

The governing equation is $x^2+y^2=25$, whose implicit derivative is $2x\cdot x'+2y\cdot y'=0$, whch gives $xx'=-yy'$. We know that $x=4$ at the moment in question, so we can determine that $y=3$. We also know that $x'=5$, so we can write $4\cdot 5=-3y'$, getting $y'=-\frac{20}{3}$, for an answer of $\frac{20}{3}$.

98. Evaluate: $\int_{-1}^{2} 3(4j)^5 dj$

$$\int_{-1}^{2} 3\cdot 1024 j^5 dj$$
$$512j^6|_{-1}^{2}$$
$$512(64-1)$$
$$32256$$

99. Evaluate: $\int_{4}^{7} 4h\sqrt{h-3}\,dh$

Substituting $u = h - 3$ means that $h = u + 3$ and $dh = du$, so that we can write

$$4\int_1^4 (u+3)u^{\frac{1}{2}}du$$
$$4\int_1^4 \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\right)du$$
$$4\left[\frac{2}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}}\right]_1^4$$
$$4\left(\frac{2}{5}(32-1) + 2(8-1)\right)$$
$$4\left(\frac{62}{5} + 14\right)$$
$$\frac{528}{5}$$

100. What is the volume of the solid generated when the area between the graphs of $y = x$ and $y = x^2 - 6$ is rotated about the line $y = 4$?

The points of intersection of the two graphs are $(3,3)$ and $(-2,-2)$, so the answer will be

$$\pi\int_{-2}^{3} ((10-x^2)^2 - (4-x)^2)dx$$
$$\pi\int_{-2}^{3} (x^4 - 21x^2 + 8x + 84)dx$$
$$\pi\left[\frac{x^5}{5} - 7x^3 + 4x^2 + 84x\right]_{-2}^{3}$$
$$\pi\left(\frac{243+32}{5} - 7(27+8) + 4(9-4) + 84(3+2)\right)$$
$$\pi(55 - 245 + 20 + 420)$$
$$250\pi$$