2015 Team Scramble
Solutions

1. **Evaluate: $\mathbf{57908 - 7039}$**

   The standard algorithm gives 50,869.

2. **What number is 3469 less than three times 8523?**

   $3 \times 8523 - 3469 = 25569 - 3469 = 22{,}100$

3. **Evaluate: $\mathbf{-2(-6 - 8(-9) - 7) - (-3)(-4)(-5) - 8(-9)}$**

   $$-2(-6 - 8(-9) - 7) - (-3)(-4)(-5) - 8(-9) = -2(-6 + 72 - 7) + 60 + 72$$
   $$= -2 \times 59 + 132 = -118 + 132 = 14$$

4. **Express 436,890,789.45 in scientific notation rounded to four significant figures.**

   Four significant figures will be 4369 (the 8 rounds up because of the 9), and there are 8 digits between the 4 and the decimal point, for an answer of 4.369×10^8.

5. **Evaluate: $\mathbf{\frac{27^5 \times 3^6}{9^4 \times 81^3}}$**

   $\frac{27^5 \times 3^6}{9^4 \times 81^3} = \frac{(3^3)^5 \times 3^6}{(3^2)^4 \times (3^4)^3} = \frac{3^{15} \times 3^6}{3^8 \times 3^{12}} = \frac{3^{21}}{3^{20}} = 3^1 = 3$

6. **Simplify by rationalizing the denominator: $\mathbf{\frac{18}{5+\sqrt{19}}}$**

   $\frac{18}{5+\sqrt{19}} = \frac{18}{5+\sqrt{19}} \times \frac{5-\sqrt{19}}{5-\sqrt{19}} = \frac{18(5-\sqrt{19})}{25-19} = 3(5 - \sqrt{19}) = 15 - 3\sqrt{19}$

7. **What is the sum of the number of vertices on an icosahedron, the number of yards in a mile, and the number of hours in a week?**

   An icosahedron has 20 faces, each of which are equilateral triangles. That would be $3 \times 20 = 60$ vertices, but each vertex is shared by five triangles, so there are really only $\frac{60}{5} = 12$ vertices. There are 1760 yards in a mile, and $7 \times 24 = 168$ hours in a week, for a total of $1760 + 168 + 12 = 1760 + 180 = 1{,}940$.

8. **Arrange the variables $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ in descending order.**
   $\mathbf{A = 7890 + 234\ B = 8479790 \div 34\ \ C = 8!\ D = 39 \times 13 + 5 \times 27}$

   $A = 8124$, $B \cong 8400000 \div 35 = 240{,}000$, $C = 40{,}320$, and $D \cong 400 + 150 = 550$, for an answer of BCAD.

9. **Simplify: $\mathbf{f + 2(3f - 4) + (5f + 6)(7 - 8f)}$**

   $$f + 2(3f - 4) + (5f + 6)(7 - 8f) = f + 6f - 8 + 35f - 40f^2 + 42 - 48f$$
   $$= -40f^2 - 6f + 34$$

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10. What value(s) of c satisfy $2(3c+4)+5(6-c)=123$?

$6c+8+30-5c=123$ becomes $c=123-38=85$.

11. What value(s) of g satisfy $3g^2-7g+12=18$?

$3g^2-7g-6=0$ factors to $(3g+2)(g-3)=0$, with roots of $-\frac{2}{3}$ and 3.

12. Miss Higgy and Hermit simultaneously see one another when they are 180 m apart. If Miss Higgy runs towards Hermit at 32 m/s, and Hermit runs away at 23 m/s, how many seconds will it take Miss Higgy to catch Hermit?

Miss Higgy is gaining at $32-23=9$ m/s, so will reach Hermit in $\frac{180}{9}=20$ seconds.

13. What are the coordinates, in the form (x,y), of the x-intercept(s) of the line $4x-7y=112$?

The x-intercept will be where $y=0$, so we can write $4x=112$, which gives $x=28$ and an answer of $(28{,}0)$.

14. What are the coordinates, in the form (x,y), of the midpoint of the line segment from $(81,234)$ to $(-123,98)$?

The x-coordinate will be $\frac{81-123}{2}=-\frac{42}{2}=-21$, and the y-coordinate will be $\frac{234+98}{2}=\frac{332}{2}=166$ for an answer of $(-21{,}166)$.

15. What is the equation of the axis of symmetry of a parabola of the form $y=ax^2+bx+c$ with a vertex at $(4,-7)$ and passing through the point $(-5,6)$?

The axix of symmetry passes through the vertex, so will be $x=4$.

16. What are the coordinates, in the form (x,y), of the vertex of the parabola with equation $x=3y^2+42y-795$?

The axis of symmetry is $y=-\frac{b}{2a}=-\frac{42}{2\times 3}=-\frac{42}{6}=-7$, for a corresponding x-value of $x=3(-7)^2+42(-7)-795=147-294-795=-147-795=-942$ and an answer of $(-942,-7)$.

17. What are the coordinates, in the form (x,y), of the y-intercept(s) of the parabola with equation $y=2x^2-7x-13$?

The y-intercept will be when $x=0$, giving $y=-13$ for an answer of $(0,-13)$.

18. A pasture contains cows, chickens, and farmhands (humans). If there are a total of 100 heads, 320 feet, and 40 hands, how many chickens are there?

Only humans have hands, so there must be $\frac{40}{2} = 20$ of them, accounting for $20 \times 2 = 40$ feet, leaving $100 - 20 = 80$ heads and $320 - 40 = 280$ feet. If all 80 heads were from cows, then there would be $4 \times 80 = 320$ feet, which is $320 - 280 = 40$ too many, so we need to convert cows to chickens, losing 2 feet each time we do. To lose 40 feet, we'll need to convert $\frac{40}{2} = 20$ cows to chickens.

19. What is the area, in square meters, of a right triangle with a leg measuring 17 m and a hypotenuse measuring 27 m?

The other leg will be $\sqrt{27^2 - 17^2} = \sqrt{729 - 289} = \sqrt{440} = 2\sqrt{110}$, so the area will be $A = \frac{1}{2}bh = \frac{1}{2} \cdot 17 \cdot 2\sqrt{110} = 17\sqrt{110}$.

20. What is the perimeter, in meters in simplest radical form, of an equilateral triangle with an area of 6 m^2?

$A = 6 = \frac{s^2\sqrt{3}}{4}$ becomes $24 = s^2\sqrt{3}$, then $\frac{24}{\sqrt{3}} = \frac{24\sqrt{3}}{3} = 8\sqrt{3} = s^2$, giving $s = \sqrt{8\sqrt{3}} = \sqrt{\sqrt{8^2 \cdot 3}} = \sqrt[4]{64 \cdot 3} = \sqrt[4]{192} = \sqrt[4]{16 \cdot 12} = 2\sqrt[4]{12}$, so that the perimeter is $3 \cdot 2\sqrt[4]{12} = 6\sqrt[4]{12}$.

21. A triangle has two sides measuring 7890 m and 245 m. What is the shortest possible integer number of meters that could be the length of its third side?

A third side measuring $7890 - 245 = 7645$ would make a degenerate triangle with no area, so we need a slightly longer side of $7645 + 1 = 7{,}646$.

22. A rhombus has sides measuring 8 m. If one of the diagonals has a length of 8 m, what is the length, in meters, of its other diagonal?

The rhombus is two equilateral triangles sharing a side (the given diagonal), so its other diagonal will be two altitudes added together. The altitude will be the long leg of a 30-60-90 triangle with a short side of $\frac{8}{2} = 4$, so the answer will be $2 \times 4\sqrt{3} = 8\sqrt{3}$.

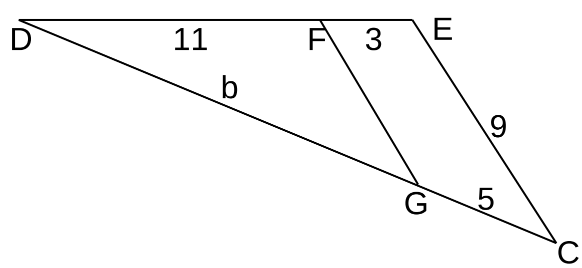
23. What is the name for a polygon with five sides?

You just have to memorize that this is a pentagon.

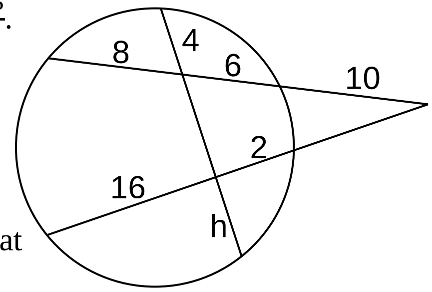
24. What is the volume, in cubic meters, of a cube with a surface area of 294 m^2?

The surface area is six squares, so each square must have an area of $\frac{294}{6} = 49$, so that the edges must be $\sqrt{49} = 7$, making the volume of the cube $7^3 = 49 \cdot 7 = 343$.

25. In ΔCDE in the figure to the right, $\overline{FG} || \overline{EC}$, and the measures of many line segments are given in meters. What is the value of b?



$\Delta DFG \sim \Delta DEC$, so we can write $\frac{b}{11} = \frac{b+5}{14}$. Cross-multiplying gives $14b = 11b + 55$, which becomes $3b = 55$, giving $b = \frac{55}{3}$.



26. The figure to the right shows a circle with two secants and a chord, with many line segments labeled in meters. What is the value of h?

The upper chord intersection gives $4x = 6 \cdot 8 = 48$, so that $x = 12$. The lower chord intersection then gives $h(16 - h) = 2 \cdot 16 = 32$, which becomes $16h - h^2 = 32$, then $h^2 - 16h + 32 = 0$, which has roots of $h = \frac{16 \pm \sqrt{16^2 - 4 \cdot 32}}{2} = 8 \pm 2\sqrt{16 - 8} = 8 \pm 4\sqrt{2}$. If h were the larger of the two, the other side would be the smaller of the two, which is less than 4 and thus violates the figure, so $h = 8 - 4\sqrt{2}$.

27. A triangle has sides measuring 9 m, 11 m, and 14 m. What is the length, in meters, of the altitude to its shortest side?

Heron's Formula gives the area of the triangle as $\sqrt{17 \cdot 8 \cdot 6 \cdot 3} = 2 \cdot 3 \cdot 2\sqrt{17} = 12\sqrt{17}$, which should also equal $\frac{1}{2} \cdot 9h$, so $h = \frac{2}{9} \cdot 12\sqrt{17} = \frac{24\sqrt{17}}{9} = \frac{8\sqrt{17}}{3}$.

28. How many diagonals can be drawn in a convex nonagon?

Any of the nine vertices could have a diagonal drawn to six other vertices (not itself, not its neighbors), for a total of $9 \times 6 = 54$ diagonals. However, each diagonal has been counted twice, once for each of its endpoints, so there are actually only $\frac{54}{2} = 27$.

29. Evaluate: $9i^7 - 6i^5 + 4i^4 - 3i^2 - 11i + 23$

$$9i^7 - 6i^5 + 4i^4 - 3i^2 - 11i + 23 = 9(-i) - 6i + 4 \cdot 1 - 3(-1) - 11i + 23$$
$$= -9i - 6i + 4 + 3 - 11i + 23 = -26i + 30$$

30. Evaluate: $\log_9 25 \times \log_5 27$

$\log_9 25 \times \log_5 27 = \frac{\ln 25}{\ln 9} \times \frac{\ln 27}{\ln 5} = \frac{2 \ln 5}{2 \ln 3} \times \frac{3 \ln 3}{\ln 5} = 3$

31. If $k(m) = 3m - 4$ and $n(p) = \frac{840}{p}$, evaluate $k\left(n\left(k(n(280))\right)\right)$.

$k\left(n\left(k(n(280))\right)\right) = k\left(n(k(3))\right) = k(n(5)) = k(168) = 3 \cdot 168 - 4 = 504 - 4 = 500.$

32. When $\left(2x - \frac{3}{x}\right)^5$ is expanded and like terms are combined, what is the coefficient of the x term?

The x term will result from three $2x$'s and two $\left(-\frac{3}{x}\right)$'s, and there are $5c3 = \frac{5!}{3!\cdot 2!} = \frac{5\cdot 4}{2} = 10$ ways to produce such a term, for an answer of $10(2x)^3\left(-\frac{3}{x}\right)^2 = 10\cdot 8x^3\cdot\frac{9}{x^2} = 720x$, for an answer of 720.

33. Express the base 6 number 5235_6 as a base 10 number.

$5235_6 = 5\cdot 6^3 + 2\cdot 6^2 + 3\cdot 6 + 5 = 5\cdot 216 + 2\cdot 36 + 18 + 5 = 1080 + 72 + 23 = 1{,}175$

34. Express the base 10 number 6792_{10} as a base 8 number.

In base 8, the digits from right to left represent $8^0 = 1$, $8^1 = 8$, $8^2 = 64$, $8^3 = 512$, $8^4 = 4096$, etc. There is one 4096 in 6792, leaving $6792 - 4096 = 2696$. There are five 512's in 2696, leaving $2696 - 5\times 512 = 2696 - 2560 = 136$. There are two 64's in 136, leaving $136 - 2\cdot 64 = 136 - 128 = 8$, which is one 8 and no 1's, for an answer of 15,210.

35. When Mrs. Math arranges her students into teams of 3, there are only two people for the last team. When she arranges them into teams of 7, there are also two people for the last team. When she arranges them into teams of 10, there are only three people for the last team. What is the smallest number of students that could be in her class?

The number of students must be of the form $3b + 2$, $7c + 2$, and $10d + 3$. The first two can be combined as $21f + 2$, which could have values of 23, 44, 65, etc. The first in that list is of the form $10d + 3$, so our answer is 23.

36. How many positive integers are factors of 834?

$834 = 2\cdot 417 = 2\cdot 3\cdot 139$, and each factor of 834 can either have or not have (two choices) each of those prime factors, making the number of factors $2^3 = 8$.

37. What is the sum of the positive integer factors of 180?

$180 = 2^2\cdot 3^2\cdot 5$, so the sum of all its factors is $(1+2+4)(1+3+9)(1+5) = 7\cdot 13\cdot 6 = 91\cdot 6 = 546$.

38. What is the 23rd term of an arithmetic sequence with first term 4579 and common difference 35?

The 23rd term will be $23 - 1 = 22$ differences away from the first term, for an answer of $4579 + 22\times 35 = 4579 + 770 = 5{,}349$.

39. What is the fifth term of a geometric sequence with first term 729 and common ratio $\frac{2}{3}$?

The fifth term will be $5 - 1 = 4$ ratios away from the first term, for an answer of $729\left(\frac{2}{3}\right)^4 = \frac{729\cdot 16}{81} = 9\cdot 16 = 144$.

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40. What is the next term of the sequence beginning 1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193?

This is a super-Fibonacci sequence, where each term is the sum of the three preceding terms, giving an answer of $57 + 105 + 193 = 250 + 105 = 355$.

41. What is the sum of the positive integers less than 32?

Using out pairs, there will be $\frac{31}{2}$ pairs, each with a sum of $1 + 31 = 32$, for an answer of $\frac{31}{2} \cdot 32 = 31 \cdot 16 = 496$.

42. The probability that I eat popcorn tonight is $\frac{3}{5}$, and the probability that I watch a movie tonight is $\frac{2}{7}$. If these events are independent, what is the probability that I eat popcorn but do not watch a movie?

$\frac{3}{5} \times \frac{5}{7} = \frac{3}{7}$

43. In how many ways can the letters in the word "TATTLETALE" be arranged?

There are ten letters overall, but there are repetitions: 4 T's, 2 A's, 2 L's, and 2 E's. This lets us write $\frac{10!}{4! \cdot 2! \cdot 2! \cdot 2!} = 10 \cdot 9 \cdot 7 \cdot 6 \cdot 5 = 630 \cdot 30 = 18{,}900$.

44. When 85 people were surveyed, 48 liked Thing 1 and 76 liked Thing 2. If 2 people liked neither, how many liked both?

$85 - 2 = 83$ people liked at least one Thing, so $83 - 76 = 7$ must have liked **only** Thing 1, so that $48 - 7 = 41$ must have liked both Things.

45. At a new casino, they charge \$10 for you to roll a standard six-sided die one time. For a 2, 3, 4, 5, or 6 they'll pay you a number of dollars equal to the square of the number you rolled. For a 1, however, you get nothing. What is the expected value of your profit (positive) or loss (negative) if you play this game one time?

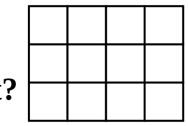
There is a $\frac{1}{6}$ chance we win nothing, a $\frac{1}{6}$ chance we win $2^2 = 4$, a $\frac{1}{6}$ chance we win $3^2 = 9$, a $\frac{1}{6}$ chance we win $4^2 = 16$, a $\frac{1}{6}$ chance we win $5^2 = 25$, and a $\frac{1}{6}$ chance we win $6^2 = 36$, so that our expected winnings are $\frac{0+4+9+16+25+36}{6} = \frac{90}{6} = 15$ dollars. Because we always spend ten dollars, our expected profit is $15 - 10 = 5$.

46. What is the mean of the data set $\{5, 93, 5, 7, 10\}$?

$\frac{5+93+5+7+10}{5} = \frac{120}{5} = 24$

47. Set K is the set of all positive two-digit multiples of 4 and Set T is the set of multiples of 3 greater than 50. How many elements does the set $K \cap T'$ have?

$K \cap T'$ will have all the elements from Set K that are not in Set T. Set K has multiples of 4 from 12 (3 4's) to 96 (24 4's), for a total of $24 - 3 + 1 = 22$ elements. $K \cap T'$ will have all of these except those that are in Set T (multiples of 3 greater than 50). The excluded elements will be multiples of both three and four, so they'll be multiples of 12, but they have to be greater than 50, so only 60, 72, 84, and 96 are excluded, making our answer $22 - 4 = 18$.



48. How many squares of any size are in the array of unit squares to the right?

There are $3 \cdot 4 = 12$ 1x1 squares, $2 \cdot 3 = 6$ 2x2 squares, and $1 \cdot 2 = 2$ 3x3 squares, for a total of $12 + 6 + 2 = 20$ squares.

49. If $\sin u = \frac{1}{6}$ and u is an angle in the second quadrant, what is the value of $\cos 2u$?

$\cos 2u = 1 - 2\sin^2 u = 1 - 2\left(\frac{1}{6}\right)^2 = 1 - \frac{2}{36} = 1 - \frac{1}{18} = \frac{17}{18}$

50. Evaluate in radians: $\cos^{-1}\left(-\frac{1}{2}\right)$

We want the angle between 0 and π that has a cosine of $-\frac{1}{2}$. If you know your unit circle, it shouldn't take too long to come up with $\frac{2\pi}{3}$.

51. Express $12e^{\frac{\pi i}{3}}$ in standard form.

$12e^{\frac{\pi i}{3}} = 12\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 12\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = 6 + 6i\sqrt{3}$

52. If $s(t) = (t+1)^2(t-1)^3$, evaluate $s'(2)$.

The product rule gives $s'(t) = 2(t+1)(t-1)^3 + (t+1)^2 \cdot 3(t-1)^2$, so $s'(2) = 2(2+1)(2-1)^3 + (2+1)^2 \cdot 3(2-1)^2 = 2 \cdot 3 \cdot 1^3 + 3^2 \cdot 3 \cdot 1^2 = 6 + 27 = 33$.

53. Evaluate: $\int_2^4 n(n-1)dn$

$\int_2^4 n(n-1)dn = \int_2^4 (n^2 - n)dn = \frac{1}{3}n^3 - \frac{1}{2}n^2\Big|_2^4 = \frac{64}{3} - \frac{16}{2} - \left(\frac{8}{3} - \frac{4}{2}\right) = \frac{56}{3} - \frac{12}{2} = \frac{56}{3} - 6 = \frac{56}{3} - \frac{18}{3} = \frac{38}{3}$

54. Evaluate: 348×679

The standard algorithm gives 236,292.

55. Evaluate: $39^3 + 41^3$

$39^3 + 41^3 = (39 + 41)(39^2 - 39 \cdot 41 + 41^2) = 80((39 - 41)^2 + 39 \cdot 41) = 80\left(2^2 + (40^2 - 1^2)\right) = 80 \cdot 1603 = 128{,}240$

56. Express $1.2\overline{34}$ as a fraction.

If $x = 1.2\overline{34}$, then $10x = 12.\overline{34}$ and $1000x = 1234.\overline{34}$. Subtracting the latter two gives $990x = 1222$, so that $x = \frac{1222}{990} = \frac{611}{495}$.

57. Hermione and Ivan both bid on a job building a brick wall. Hermione knows she could do it in 12 hours, and Ivan knows he could do it in 16 hours. The customer decides to hire them both, and it turns out that they work together so well that they lay a total of 60 more bricks each hour than they would if they were working simultaneously but separately. If the job requires 4800 bricks, how many minutes (to the nearest minute) does it take the two of them to finish the job?

Hermione's speed must be $\frac{4800}{12} = 400$ bricks per hour, while Ivans's must be $\frac{4800}{16} = 300$. Thus, together they'll work at $400 + 300 + 60 = 760$ bricks per hour, so that the job would take $\frac{4800}{760} = \frac{480}{76} = \frac{120}{19}$ hours, which is $\frac{120 \times 60}{19} = \frac{7200}{19} \cong 379$ minutes.

58. Brooks bikes to work at a speed of 25 km per hour, and later bikes home along the same route at a speed of 20 km per hour. What is his average speed, in kilometers per hour, for the entire roundtrip?

We don't know how long his trips are, so let's call it d. We can then know that his times were $\frac{d}{25}$ and $\frac{d}{20}$, so that his average speed is $\frac{d+d}{\frac{d}{25}+\frac{d}{20}} = \frac{2d}{\frac{9d}{100}} = \frac{200}{9}$.

59. What is the shortest distance from the point $(38, -9)$ to the line $y = -\frac{4}{3}x + 5$?

The line can also be written $4x + 3y - 15 = 0$, so that the distance will be $\frac{|4 \cdot 38 + 3(-9) - 15|}{\sqrt{4^2 + 3^2}} = \frac{|152 - 27 - 15|}{\sqrt{16+9}} = \frac{|110|}{\sqrt{25}} = \frac{110}{5} = 22$.

60. The point $(-68, 94)$ is rotated $2790°$ clockwise about the point $(341, 89)$ to point J, which is then reflected across the line $x + y = 249$ to point K, which is then rotated $5760°$ counter-clockwise about the point $(16, -858)$ to point L. What are the coordinates, in the form (x, y) of point L?

$2790°$ clockwise is $31 \cdot 90°$ clockwise, which is the same as $90°$ counter-clockwise. $(-68{,}94)$ is $341 - (-68) = 409$ to the left of $(341{,}89)$ and $94 - 89 = 5$ units above it. After a counter-clockwise rotation, the point J will be 409 below and 5 to the left, which is $(-320{,}336)$. Drawing a square with $x + y = 249$ as a diagonal shows that point K will be $(569, -87)$. $5760° = 64 \cdot 90°$, which is the same as not rotating at all, so point L will also be $(569, -87)$.

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61. When the digits of a positive four-digit integer are reversed, a new positive four-digit integer is created, and the positive difference between the two is M. How many possible values might M have?

When ABCD is reversed to be DCBA, the difference is $999(A-D)+90(B-C)$. $A-D$ can range from 0 to 8 (9 choices) and $B-C$ can range from -9 to 9 (19 choices) for a total of $9 \times 19 = 171$ possibilities. However, $B-C$ must be positive if $A-D$ is zero, so there are ten choices (-9 to 0) that are not allowed, for an answer of $171-10=161$. Note that 90 and 999 cannot combine to eliminate options unless $B-C$ becomes as large as 111.

62. If you can buy L liters of pudding for D dimes, how many quarters would be needed to buy 40 liters of pudding?

Pudding apparently costs $\frac{10D}{L}$ cents per liter, so 40 liters would cost $\frac{400D}{L}$ cents, which would be $\frac{400d}{25L}=\frac{16D}{L}$. Because this might not be an integer number of quarters, it would be best to write $\left\lceil\frac{16D}{L}\right\rceil$, but that's not required to get credit for this problem.

63. Express the solution to the system of equations $m+n+p=11$, $m+n+q=-8$, $m+p+q=24$, and $n+p+q=-12$ in the form (m,n,p,q).

If we add all these equations, we get $3m+3n+3p+3q=15$, which becomes $m+n+p+q=5$. Subtracting each original equation from this one gives $q=-6$, $p=13$, $n=-19$, and $m=17$, for an answer of $(17,-19,13,-6)$.

64. What is the area, in square meters, of an isosceles triangle with sides measuring 22 m and 61 m?

You can't have a 22-22-61 triangle, so it must be a 22-61-61 triangle. Drawing a median/altitude to the short side produces two right triangles with a leg of 11 and a hypotenuse of 61. The common leg of these triangles is $\sqrt{61^2-11^2}=\sqrt{(61-11)(61+11)}=\sqrt{50\cdot 72}=\sqrt{3600}=60$, so that the area of the original triangle is $\frac{1}{2}\cdot 22\cdot 60=11\times 60=660$.

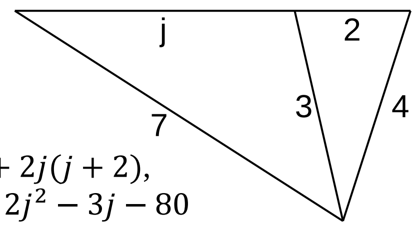
65. What is the minimum perimeter, in meters, of a parallelogram with two sides measuring 24 m and an area of 912 m^2?

The height of the parallelogram perpendicular to the sides measuring 24 must be $\frac{912}{24}=\frac{456}{12}=\frac{228}{6}=\frac{114}{3}=38$. Given that, the two sides measuring 24 might be directly above one another or skewed laterally a little or a lot. Thinking about it, the other two sides get longer the more skewed the two 24 sides are, so we'd like them directly above one another, making the parallelogram a rectangle. This makes the other sides 38, so that the perimeter is $24+24+38+38=48+76=124$.

66. What is the volume, in cubic meters, of an octahedron with edges measuring 8 m?

An octahedron can be built from eight sixths of a cube joined by their right-angled vertices at its center. The cubes the pieces were cut from had face diagonals of 8, so their edges must have been $\frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$. The volume would then be $\frac{8}{6}\left(4\sqrt{2}\right)^3 = \frac{4}{3} \cdot 128\sqrt{2} = \frac{512\sqrt{2}}{3}$.

67. The figure to the right shows a triangle with a cevian, with all line segments labeled in meters. What is the value of *j*?

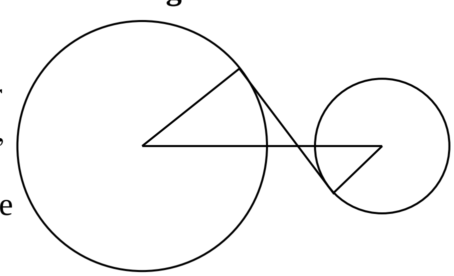


Stewart's Theorem lets us write $7^2 \cdot 2 + 4^2 j = 3^2(j+2) + 2j(j+2)$, which becomes $98 + 16j = 9j + 18 + 2j^2 + 4j$, then $0 = 2j^2 - 3j - 80$ with roots of $\frac{3 \pm \sqrt{3^2+640}}{2 \cdot 2} = \frac{3 \pm \sqrt{649}}{4}$, only one of which is positive..

68. A rectangle has sides measuring 2 m and 3 m. What is the largest area, in square meters, that can be covered by two non-overlapping circles that do not extend outside the rectangle? The circles do not necessarily have to be congruent.

If one circle were inscribed, it could have a radius of 1 m. If another circle of radius r were inscribed in a far corner of the rectangle until it touches the 1 m circle, radii can be drawn along with a line parallel to a side to create a right triangle with legs of $1 - r$ and $2 - r$ and a hypotenuse of $1 + r$. We can then write $(1-r)^2 + (2-r)^2 = (1+r)^2$, which becomes $1 - 2r + r^2 + 4 - 4r + r^2 = 1 + 2r + r^2$, then $r^2 - 8r + 4 = 0$, which has roots of $\frac{8 \pm \sqrt{8^2 - 4 \cdot 4}}{2} = 4 \pm \sqrt{16-4} = 4 \pm \sqrt{12} = 4 \pm 2\sqrt{3}$, only the smaller of which could fit. This would give a total area of $1^2\pi + \left(4 - 2\sqrt{3}\right)^2\pi = \pi + \left(16 - 16\sqrt{3} + 12\right)\pi = \left(29 - 16\sqrt{3}\right)\pi$. If the circle with a radius of 1 m had its radius get a little smaller, the other circle's radius would get a little larger, but the annulus of area lost from the bigger circle would have a much larger area than that of the smaller circle, so our answer is the largest possible.

69. Two circles with radii of 13 m and 16 m have their centers 30 meters apart. What is the length, in meters, of the shortest line segment that can be drawn tangent to both circles?



A rough sketch shows that the circles are only one meter apart, so that their internal tangent will be shorter than their external tangent. The segment between the circles' centers, the internal tangent, and the radii to the internal tangent form two similar right triangles as shown in the figure to the right. Rotating one of the triangles around the center of its hypotenuse allows you to see an even larger right triangle with a hypotenuse of 30 and one leg measuring $13 + 16 = 29$, so that the other leg (the internal tangent) will have a length of $\sqrt{30^2 - 29^2} = \sqrt{900 - 841} = \sqrt{59}$.

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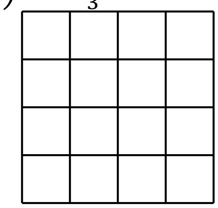
70. A cow is tied to the exterior of the $60°$ corner of a barn in the shape of a 30-60-90 triangle with a short leg measuring 12 m. If the rope has a length of 28 m, what is the area, in square meters, of the grass that the cow can graze? Note: the barn is closed; the cow cannot go inside.

Clearly, the cow can graze $\frac{5}{6}$ of a circle with a radius of 28. In addition, it can graze $\frac{1}{4}$ of a circle with a radius of $28 - 12 = 16$. Finally, the cow can graze $\frac{5}{12}$ of a circle with a radius of $28 - 24 = 4$. The other leg measures $12\sqrt{3} > 12 \cdot 1.7 = 12 + 8.4 = 20.4$, which is longer than $16 + 4 = 20$, so none of these circular regions overlaps, and our answer will be $\frac{5}{6} \cdot 28^2\pi + \frac{1}{4} \cdot 16^2\pi + \frac{5}{12} \cdot 4^2\pi = \pi\left(\frac{5 \cdot 14 \cdot 28}{3} + 4 \cdot 16 + \frac{5 \cdot 4}{3}\right) = \pi\left(\frac{1960+192+20}{3}\right) = \frac{2172\pi}{3} = 724\pi$.

71. In how many distinguishable ways can congruent rectangles with sides measuring 1 m and 2 m be used to tile a square with sides measuring 4 m? Note that the square could be rotated.



Consider the 4x4 grid to the right. Each rectangle could either go vertically or horizontally. If a tile goes horizontally, there must be another horizontal tile in the same columns. Thinking about it, there is one way to tile the square using only vertical rectangles, and this way is the same as using only horizontal tiles, because the square could be rotated. There is no way to use just one horizontal tile, but we could use two. There turn out to be five ways to do this, once we account for $180°$ rotations. There is no way to use three horizontal rectangles, but we could use four, and this is the most we need to consider, because using six would be analogous to using two, due to rotations. Analyzing 4 & 4 is a little harder than 2 & 6, because $90°$ rotations must be considered, in addition to $180°$ rotations. There are six of these, for an answer of $1 + 5 + 6 = 12$.

72. A right rectangular prism made from black plastic has edges measuring 12 m, 4.5 m, 9 m. It is painted blue on all sides, then cut into the smallest possible number of congruent cubes. How many of the resulting cubes have more black faces than blue faces?

This prism could be cut into 1.5 m cubes, which means 8 slices in the first direction, 3 in the second, and 6 in the third, for a total of $8 \cdot 3 \cdot 6 = 24 \cdot 6 = 144$ cubes. Of these, some are in the interior and show six black faces, some are on faces and show five black faces and one blue face, some are on edges and show four black and two blue faces, and come are on vertices and show three black and three blue faces. Of these, only the last do not have more black faces than blue faces. There are 8 vertices, so our answer is $144 - 8 = 136$.

73. What are the coordinates, in the form (x, y) of the upper focus of the conic section with equation $-4x^2 + 24x + 9y^2 + 18y = 243$?

Completing the squares gives $-4(x-3)^2 + 9(y+1)^2 = 243 - 36 + 9 = 216$, which then becomes $-\frac{(x-3)^2}{54} + \frac{(y+1)^2}{24} = 1$. The center is at $(3, -1)$ and the focal distance is $\sqrt{54 + 24} = \sqrt{78}$. The foci are in the y-direction, so the upper one will be at $\left(3, \sqrt{78} - 1\right)$.

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74. What is the area of the ellipse with equation $x^2 + 10x + 4y^2 - 16y = 359$?

Completing the square gives $(x+5)^2 + 4(y-2)^2 = 359 + 25 + 16 = 400$, which becomes $\frac{(x+5)^2}{400} + \frac{(y-2)^2}{100} = 1$, with semi-axes of $\sqrt{400} = 20$ and $\sqrt{100} = 10$, for an area of $A = \pi ab = \pi \cdot 20 \cdot 10 = 200\pi$.

75. What value(s) of j satisfy $3^{2j} + 729 = 3^{j+4} + 3^{j+2}$?

We intended this to be a "hidden quadratic", but think it solves easier if we just put everything as exponents of 3: $3^{2j} + 3^6 = 3^{j+4} + 3^{j+2}$. Now it seems fairly easy to find $j = 2$ and $j = 4$ by inspection.

76. q varies jointly with the square of r and the square root of t. If $q = 36$ when $r = t = 36$, what value of r will cause q to equal 216 when $t = 144$?

t went from 36 to 144, which means it was multiplied by $\frac{144}{36} = 4$, so q should have been multiplied by $\sqrt{4} = 2$ to become $36 \cdot 2 = 72$. Instead, $q = 216$, which is $3 \cdot 72$, so r must have been multiplied by $\sqrt{3}$ to become $36\sqrt{3}$.

77. The vertex and x-intercepts of the parabola with equation $y = 2x^2 - x - 15$ are used as the vertices of a triangle. What is the area of the triangle?

The vertex is on the axis of symmetry, which is $x = -\frac{b}{2a} = \frac{1}{2\cdot 2} = \frac{1}{4}$, so its y-coordinate will be $y = 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} - 15 = -15\frac{1}{8} = -\frac{121}{8}$, which will be the "height" of the triangle. To find the x-intercepts, we'll factor it to get $0 = (2x+5)(x-3)$, getting roots of $-\frac{5}{2}$ and 3. The difference between these is $3 - \left(-\frac{5}{2}\right) = \frac{11}{2}$, so the area will be $A = \frac{1}{2}bh = \frac{1}{2} \cdot \frac{11}{2} \cdot \frac{121}{8} = \frac{1331}{32}$.

78. What is the product of the three complex cube-roots of -27?

We can write $z^3 = -27$, which becomes $z^3 + 27 = 0$. The product of the roots of this equation is $(-1)^3 \cdot 27 = -27$.

79. What is the least common multiple of 548 and 458?

$548 = 2^2 \cdot 137$ and $458 = 2 \cdot 229$, and even if 229 has prime factors, 137 won't be one of them, so the LCM will be $229 \cdot 548 = 137 \cdot 2 \cdot 458 = 125{,}492$.

80. 1 m unit cubes are used to build a block measuring 4 cm by 8 cm by 6 cm. A tiny ant then chews his way in a straight line from one vertex of the block to the furthest vertex. How many cubes does the ant pass through? The ant is so tiny that he does not "pass through" cubes if he is merely passing through where their edges or vertices meet.

The ant's path can be considered two sections that each go diagonally through a 2x4x3 set of cubes, so we can solve that problem and just double our answer. Consider a lateral 2x4 grid with a diagonal drawn through it; the ant will transition between cubes three times, as $\frac{1}{4}$ of the way, $\frac{1}{2}$ of the way, and $\frac{3}{4}$ of the way, so he'll pass through four cubes (counting the cube he starts in). However, he's also traveling up through three layers of cubes, and he'll transition between cubes at $\frac{1}{3}$ of the way through and $\frac{2}{3}$ of the way through, neither of which duplicates a transition we already counted, so we'll add two new cubes to get $4 + 2 = 6$ for one 2x4x3 set of cubes, and a total answer of $2 \times 6 = 12$.

81. The palindromes larger than 12345 are written in a list in ascending order. What is the 111th member of this list?

The first palindromes we'll encounter will be of the form $1ABA1$, and there are $10 \cdot 10 = 100$ of them. However, we've already skipped ten like $10C01$, ten like $11D11$, and four like $12E21$, so there are $100 - 24 = 76$ left, leaving $111 - 76 = 35$ left. They will be of the form $2FGF2$. The first ten will be of the form $20H02$, ten more will be like $21I12$, and ten more will be like $22I22$, so that our answer will be of the form $23J32$. We want the fifth of these, so count 23032, 23132, 23232, 23332, and 23432.

82. What is the next term of the sequence beginning with 14, 32, 40, 48, 66, 72, 92, 108, 118, and 162?

After trying several things, you'll hopefully notice that this is a sequence comprised of two sub-sequences: 14, 40, 66, 92, and 118 are an arithmetic sequence with a difference of 26, so the next term will be $118 + 26 = 144$. Just to be sure, 32, 48, 72, 108, and 162 form a geometric sequence with common ratio $\frac{3}{2}$.

83. What is the sum of the perfect squares from 100 to 400, inclusive?

We want the first 20 squares but not the first nine squares, and can use the formula ($\frac{n(n+1)(2n+1)}{6}$, getting $\frac{20 \cdot 21 \cdot 41}{6} - \frac{9 \cdot 10 \cdot 19}{6} = 2870 - 285 = 2{,}585$.

84. What is the sum of the twelve smallest positive perfect cubes?

Again, there's a formula you can memorize to make this faster, and it's pretty easy… the sum of the first n cubes is simply the square of the sum of the first n integers. In this case, that gives $\left(\frac{n(n+1)}{2}\right)^2 = \left(\frac{12 \cdot 13}{2}\right)^2 = (6 \cdot 13)^2 = 78^2 = 6{,}084$.

85. What is the sum of the positive two-digit multiples of 6 that are not written using the digit 6?

The multiples of 6 go from 12 to 96 (two 6's to 16 6's), but we don't want 36, 60, 66, or 96, so we want the sum of the first 15 multiples of 6, minus $6 + 36 + 60 + 66 = 168$, which is $6\left(\frac{15 \cdot 16}{2}\right) - 168 = 6 \cdot 15 \cdot 8 - 168 = 90 \cdot 8 - 168 = 720 - 168 = 552$.

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86. A trusted friend flips six coins and tells you that there were at least two each of heads and tails. What is the probability that there were three of each?

The result might have been HHTTTT ($6c2 = \frac{6\cdot5}{2} = 3\cdot5 = 15$ ways), HHHTTT ($6c3 = \frac{6\cdot5\cdot4}{3\cdot2} = 5\cdot4 = 20$ ways), or HHHHTT (15 ways again), so our probability is $\frac{20}{15+20+15} = \frac{20}{50} = \frac{2}{5}$.

87. Wendy and Xi love to play Flip!, in which they take turns flipping a coin and keeping track of the results. Once the same result has been flipped twice in a row (e.g. two heads in a row or two tails in a row), the player who flipped the second one is crowned the winner. What is the probability that the first player wins?

The first player wins if the dice flip ABB (2 ways, for a probability of $\frac{2}{8} = \frac{1}{4}$), ABABB (2 ways, for a probability of $\frac{2}{32} = \frac{1}{16}$), ABABABB (2 ways, for a probability of $\frac{2}{128} = \frac{1}{64}$), etc. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$ is an infinite geometric series with a sum of $S = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$.

88. Because you're so nice, Mathy Noll lets you select one of four identical gift-wrapped boxes from his Table of Gifts. He says he remembers that one of the boxes contains one million dollars, and that the others contain leftovers from various meals he's eaten, but he's not sure which is which. You figure there's not much to lose, so you touch a box with the intent of opening it. Suddenly Mathy says "I have to know!" and unwraps one of the other boxes ridiculously quickly, revealing some moldy pizza in a tupperware. "I'm sorry," he says, "you can switch your choice if you want to." If you take advantage of his offer and select one of the other two boxes (not the one you picked initially, nor the one he opened), what is the probability it contains the million dollars?

Because Mathy doesn't know where the "good present" is, we don't learn anything when he opens a "bad present". He might have opened the million dollars, it was just luck that he didn't, not knowledge like in the traditional Monty Hall problem. Because of this, we still don't know much about where the million dollars is. There are three boxes, all of which have an equal chance of containing it, so each box has a $\frac{1}{3}$ probability. When I switch boxes, my new box has a $\frac{1}{3}$ probability of containing the million dollars.

89. What is the shortest distance from the point $(1, 1, 1)$ to the plane $2x - y - 2z = 11$?

Much like with a line, we can write the plane as $2x - y - 2z - 11 = 0$, and then compute the distance to be $\frac{|2\cdot1-1-2\cdot1-11|}{\sqrt{2^2+1^2+2^2}} = \frac{|2-1-2-11|}{\sqrt{4+1+4}} = \frac{|-12|}{\sqrt{9}} = \frac{12}{3} = 4$.

90. In a seven-element data set of integer test scores from 0 to 100 inclusive, the mean is 58, the median is 68, and the range is 42. What is the largest possible value of the unique mode?

Writing the elements in ascending order, we get $x-42$, $x-41$, $x-40$, 68, 69, x, x. If the mean is 58, the sum should be $7 \cdot 58 = 406 = 5x - 123 + 137 = 5x + 14$. This becomes $5x = 392$, giving $x = 78.4$, so that the mode can be 78 so long as the $x - 40$ term is increased a little bit.

91. In the data set $\{8, 93, 452, 90, 3, 456, 8, d, f\}$, d and f are integers, and the mean is greater than the median, which is greater than the unique mode. What is the smallest possible sum of d and f?

Writing the elements in increasing order, we get 3, 8, 8, 90, 93, 452, 456, with the positions of d&f unknown. The mean would be $\frac{3+8+8+90+93+452+456+d+f}{9} = \frac{19+183+908+d+f}{9} = \frac{1110+d+f}{9} \cong 123.\overline{3} + \frac{d+f}{9}$. The median could range from 8 to 93, depending on the values of d & f. The unique mode is either 8 or d & f & x (an existing element). Because the mean is so large compared to the mode (and a possible value of the median), $d+f$ can likely be very negative. If they were both negative, the mode and median would be equal, which isn't allowed, so let's let $d = 9$ and f be negative. Now the mode is 8, the median is 9, and the mean can be $\frac{1110+9+f}{9} > 9$, giving $1119 + f > 81$ and $f > -1038$, so that $f = -1037$ and $d + f = -1028$.

92. Set J is the set of one-digit counting numbers, and Set K is the set of all counting numbers less than 20. How many sets are subsets of K, supersets of J, contain at least two prime numbers, and at most six composite numbers?

The sets in question must contain all of J, so they'll have 1-9 for sure, which specifically means they'll have the primes 2, 3, 5, and 7 and they'll have the composites 4, 6, 8, and 9. Thus, they'll definitely have at least two prime numbers, but they can only have at most two more composite numbers. Thus, each of 11, 13, 17, and 19 can either be present or not, which is $2^4 = 16$ options. Of 10, 12, 14, 15, 16, and 18 (six numbers), we can have at most two. There are $6c0 = 1$ ways to have none of them, $6c1 = 6$ ways to have one of them, and $6c2 = 15$ ways to have two of them, for a total of $15 + 6 + 1 = 22$ options and a total answer of $22 \cdot 16 = 352$.

93. When Aryc, Brynn, Candyce, and Dylan got together for game night recently, they brought their children (Myron, Nancy, Orly, and Penny), their pets (Elly, Freddy, Gryphon, and Hungry), some dessert to share (Ice Cream, Juicers, Kit Kats, and Lemon Bars). The following statements are all true:

The Lemon Bars and Penny came together, as did the Ice Cream and Freddy.
Gryphon, Myron, and the Lemon Bars all came separately, but Elly & Nancy came together.
None of Hungry, Penny, or the Juicers came with Dylan.
Aryc brought the Kit Kats, Brynn did not bring Elly, and Candyce brought Orly.

What three things did Dylan bring?

Using first letters, we're trying to associate four options in three categories with A, B, C, and D. It's easier to record associations than the lack of associations, so we'll do those first. We can write LP and IF from the first clue, and EN from the second clue. The fourth clue tells us that K is with A and O is with C. At this point, we know that P & L must be with B or D, and F & I must be with B, C, or D. Looking at the negatives in the clues, P didn't come with D, so P & L must be with B. D also doesn't have J, so it must have I (and thus F), leaving J with C. E & N now can only go with A, forcing M to be with D, which is enough to answer the question: Dylan brought Myron, Freddy, and Ice Cream.

94. All Knights speak only truths, all Knaves speak only lies, and all of the people below are one or the other.

Person Z: Y is not a knave.
Person Y: Neither Z nor W is a knave.
Person X: At least one of Y or Z is a knave.
Person W: Both of X and V are knights.
Person V: I am not a knave.

List the letters of all of the people above who must be knaves.

Frequently, the best approach to a problem like this is to just make an assumption and see how it pans out. Let's assume that Z is telling the truth (a knight). Then it's true that Y is not a knave, so Y is a knight, and will say something true, too. Thus, neither Z nor W is a knave, meaning they're both knights. This agrees with our assumption about Z, which is nice. W must be saying something true, too, so then X and V are both knights. Uh, oh, X should be saying something true, but it's not true that at least one of Y or Z is a knave; they're both knights under this scenario. That means that our initial assumption must have been wrong; Z is not a knight, so he must be a knave. Now we start over with that assumption…

If Z is a knave, then he's lying, so Y is also a knave, so he's lying and at least one of Z and W is a knave. We know that Z is a knave, so we don't really know anything about W yet; he could be either and Y would still be lying. X talks about Y & Z, and says something we know is true, so X must be a knight. So far, we think W could go either way. If he's a knight (sub-assumption), then X and V must both be knights. X *is* a knight, and V could be… I knight would truthfully say he was not a knave. So, W can be a knight. Can W be a knave? If so, he's lying when he says X & V are both knights, but since we know X is a knight, then V must not be. Can that be the case? Yes, a knave would lie and say that they were not a knave.

Thus, Z is a knave, Y is a knave, X is a knight, and W & V are either (but match each other). This makes the answer Z & Y, as they are the only ones that *must* be knaves.

95. A triangle has an angle of $120°$ between sides measuring 3 m and 4 m. What is the length, in meters, of the third side?

Using the Law of Cosines, $c^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos 120° = 9 + 16 - 24\left(-\frac{1}{2}\right) = 25 + 12 = 37$, so $c = \sqrt{37}$.

96. A triangle has sides measuring $\sqrt{10}$ m, $\sqrt{17}$ m, and $\sqrt{29}$ m. What is its area, in meters?

You can do this using Hero's formula, but it's a lot of computation. It is easier to realize that $\sqrt{10}$ is the hypotenuse of a 1x3 right triangle, $\sqrt{17}$ is for a 1x4, and $\sqrt{29}$ is for a 2x5. Could the vertices of this triangle be lattice points? Sure, if the 1 & 4 add to make a 5 and the 3 & 1 subtract to make a 2. For example, the points $(0,0)$, $(1,3)$, and $(5,2)$ do this, and give a triangle with the desired side lengths. This triangle's area is $3 \cdot 5 - \frac{1}{2}(3 \cdot 1 + 4 \cdot 1 + 5 \cdot 2) = 15 - \frac{1}{2}(3 + 4 + 10) = 15 - \frac{17}{2} = \frac{13}{2}$.

97. What is the sum of the values of n between 0 and 2π that satisfy $\sin n + 2\cos^2 n = 1$?

We can replace $2\cos^2 n = 2(1 - \sin^2 n) = 2 - 2\sin^2 n$ to get $\sin n + 2 - 2\sin^2 n = 1$, then rearrange to get $0 = 2\sin^2 n - \sin n - 1$, which factors to $(2\sin n + 1)(\sin n - 1)$, with roots of $\sin n = -\frac{1}{2}$ and $\sin n = 1$, giving $n = \frac{7\pi}{6}$ and $n = \frac{11\pi}{6}$, as well as $n = \frac{\pi}{2}$, for an answer of $\frac{7\pi}{6} + \frac{11\pi}{6} + \frac{\pi}{2} = \frac{18\pi}{6} + \frac{\pi}{2} = 3\pi + \frac{\pi}{2} = \frac{7\pi}{2}$.

98. If $q(r) = 3^{2r}$, evaluate $\frac{dq}{dr}$ when $r = -\frac{1}{2}$.

You should probably memorize the derivative of a^x, but I never have; I know there's an $\ln a$ involved. Instead, I convert to e^x, which is easy to remember. In this case, $3 = e^{\ln 3}$, so we can write $q(r) = \left(e^{\ln 3}\right)^{2r} = e^{(2\ln 3)r}$, so that $\frac{dq}{dr} = e^{(2\ln 3)r} \cdot 2\ln 3$. When $r = -\frac{1}{2}$ this becomes $\frac{dq}{dr} = e^{(2\ln 3)\left(-\frac{1}{2}\right)} \cdot 2\ln 3 = e^{-\ln 3} \cdot 2\ln 3 = \left(e^{\ln 3}\right)^{-1} \cdot 2\ln 3 = 3^{-1} \cdot 2\ln 3 = \frac{2\ln 3}{3}$.

99. Petra is using a 20 ft. ladder to spray Formula 409 on a window 19 ft. above the ground. Suddenly the base of the ladder begins to slip away from the wall! Strangely, the ladder's base slides smoothly at a speed of 2 ft. per second while the top of the ladder slides smoothly down the wall of the house. How fast, in feet per second, is the top of the ladder sliding down when the base of the ladder is 16 ft. from the house?

No matter what the ladder does, $b^2 + h^2 = 20^2 = 400$, and differentiating implicitly with respect to time gives $2bb' + 2hh' = 0$, which becomes $bb' = -hh'$. At the moment in question, $b = 16$, $b' = 2$, and we want to find h', but we need h. We can use the original relation to find that $h = \sqrt{20 - 16^2} = \sqrt{400 - 256} = \sqrt{144} = 12$, so that we can write $16 \cdot 2 = -12h'$, getting $h' = -\frac{32}{12} = -\frac{8}{3}$. Thus, the speed at which the ladding is sliding *down* is $\frac{8}{3}$.

100. What is the average value of the function $k(m) = 3m^4$ between $m = 2$ and $m = 3$?

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The average value is simply the sum of all the values divided by the number of values. Of course, in calculus, these are both infinity, but we can still use the idea to remember the formula $\frac{\int_2^3 3m^4 dm}{3-2} = \frac{\frac{3}{5}m^5\Big|_2^3}{1} = \frac{3}{5}(3^5 - 2^5) = \frac{3}{5}(243 - 32) = \frac{3}{5} \cdot 211 = \frac{633}{5}$.