2014 Team Scramble
Solutions

1. Simplify: $\sqrt{5200}$

$\sqrt{5200} = 10\sqrt{52} = 20\sqrt{13}$

2. Evaluate: $\frac{5}{6} - \frac{3}{8}$

$\frac{5}{6} - \frac{3}{8} = \frac{20}{24} - \frac{9}{24} = \frac{11}{24}$

3. Express .00077652 in scientific notation rounded to three significant figures.

$.00077652 = 7.7652 \times 10^{-4} \approx 7.77 \times 10^{-4}$

4. Evaluate: $\frac{11! \times 3!}{5! \times 7!}$

$\frac{11! \times 3!}{5! \times 7!} = \frac{11 \times 10 \times 9 \times 8}{5 \times 4} = 11 \times 9 \times 4 = 11 \times 36 = 396$

5. Evaluate: $\binom{7}{4}$

$\binom{7}{4} = \frac{7 \times 6 \times 5}{3 \times 2} = 7 \times 5 = 35$

6. Express in simplest radical form: $\sqrt[3]{875}$

$\sqrt[3]{875} = \sqrt[3]{5 \times 175} = \sqrt[3]{5 \times 5 \times 35} = \sqrt[3]{5 \times 5 \times 5 \times 7} = 5\sqrt[3]{7}$

7. Arrange the letters in decreasing numerical order:
$Z = 13 \times 17$ $\quad Y = 987 - 678$ $\quad X = 5432 \div 19$ $\quad W = 123 + 421$

$Z = 13 \times 17 = 221$, $Y = 987 - 678 = 309$, $X \approx 270 +$, and $W = 544$, making the answer WYXZ.

8. What ordered quadruple, of the form (r, s, t, u), satisfies the equations $2r + 2s + 2t + 3u = 10$, $2s + 3t + 2u + 2r = 11$, $2t + 2u + 3r + 2s = 6$, and $2u + 2r + 3s + 2t = 9$?

Adding the four equations gives $9r + 9s + 9t + 9u = 36$, so that $r + s + t + u = 4$ and $2r + 2s + 2t + 2u = 8$. Subtracting this from the original equations gives $u = 2$, $t = 3$, $r = -2$, and $s = 1$, for an answer of $(-2,1,3,2)$.

9. What value(s) of g satisfy $5(78g - 90) + 6(12 - 34g) = 16944$?

$5(78g - 90) + 6(12 - 34g) = 16944$ becomes $390g - 450 + 72 - 204g = 16944$, then $186g = 17322$, giving $g = \frac{17322}{186} = \frac{8661}{93} = \frac{2887}{31}$.

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10. Amy gives half of her marbles to Billy, then buys 47 marbles, then gives a third of her marbles to Celia, after which she gives 74 marbles to Davie, ending with 174 marbles. How many marbles did Amy start with?

If she had 174 marbles after giving Davie 74, she must have had $174 + 74 = 248$ beforehand. Prior to that she gave a third away, so she must have had $\frac{3}{2} \times 248 = 248 + 124 = 372$. This was after she bought 47 marbles, so she must have had $372 - 47 = 325$ before that, which is half of the $2 \times 325 = 650$ she started with.

11. What value(s) of c satisfy $7c + 5 = 3c - 2$?

$7c + 5 = 3c - 2$ becomes $4c = -7$, giving $c = -\frac{7}{4}$.

12. What value(s) of b satisfy $2b^2 - 5b - 12 = 0$?

$2b^2 - 5b - 12 = 0$ factors to $(2b + 3)(b - 4) = 0$ with roots of $-\frac{3}{2}$ and 4.

13. If you travel at a constant speed of 79 meters per second for one second less than a minute, how many meters do you travel?

We travel for $60 - 1 = 59$ seconds, so we travel $79 \times 59 = 80 \times 60 - 80 - 59 = 4800 - 80 - 59 = 4720 - 59 = 4661$.

14. You bike uphill to school at an average speed of 4 meters per second, but bike home at a downhill speed of 9 meters per second. What was your average speed, in meters per second, for the round-trip to and from school?

Let's say it's d meters to school, so that our total trip is $2d$, and it takes us $\frac{d}{4} + \frac{d}{9} = \frac{13d}{36}$ seconds to do it, for an average speed of $\frac{2d}{\frac{13d}{36}} = \frac{2d \times 36}{13d} = \frac{72}{13}$ meters per second.

15. Express the equation of the line through the points $(-3, 5)$ and $(1, -1)$ in slope-intercept form $(y = mx + b)$.

The slope is $\frac{5 - -1}{-3 - 1} = \frac{6}{-4} = -\frac{3}{2}$, so the line is $y = -\frac{3}{2}x + b$. Substituting the latter point gives $-1 = -\frac{3}{2} + b$, so that $b = -1 + \frac{3}{2} = \frac{1}{2}$ for an answer of $y = -\frac{3}{2}x + \frac{1}{2}$.

16. What are the coordinates, in the form (x, y), of the midpoint of the line segment connecting the points $(-9, -8)$ and $(2, -4)$?

The x-coordinate will be $\frac{-9+2}{2} = -\frac{7}{2}$, an the y-coordinate will be $\frac{-8+(-4)}{2} = -\frac{12}{2} = -6$, for an answer of $\left(-\frac{7}{2}, -6\right)$.

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17. What are the coordinates, in the form (x, y), of the point of intersection of the lines $y = 2x + 3$ and $3x - 2y = 1$?

Substituting the first equation into the second gives $3x - 2(2x + 3) = 1$, which becomes $3x - 4x - 6 = 1$, then $-x = 7$, so that $x = -7$. Substituting this value into the first equation gives $y = 2(-7) + 3 = -14 + 3 = -11$, for an answer of $(-7, -11)$.

18. The point $(-3, 7)$ is rotated $90°$ counter-clockwise about the point $(5, 2)$ to Point D. What are the coordinates, in the form (x, y), of Point D?

$(-3,7)$ is $5 - (-3) = 8$ to the left of $(5,2)$, and is $7 - 2 = 5$ above it. After rotating, it will be 8 below and 5 to the left, so its new coordinates will be $(0, -6)$.

19. What are the coordinates, in the form (x, y), of the rightmost x-intercept of the parabola $y = -3x^2 + 12x - 4$?

The quadratic formula gives roots at $x = \frac{-12 \pm \sqrt{12^2 - 4(-3)(-4)}}{2(-3)} = \frac{12 \pm \sqrt{144 - 48}}{6} = \frac{12 \pm \sqrt{96}}{6} = \frac{12 \pm 4\sqrt{6}}{6} = \frac{6 \pm 2\sqrt{6}}{3}$, for an answer of $\left(\frac{6+2\sqrt{6}}{3}, 0\right)$.

20. Mr. Jacobson asks his students to find the roots of a parabola of the form $r^2 + Br + C = 0$ on the screen. Megan miscopied the value of B and got roots of 6 and -4. Greg miscopied the value of C and got roots of -6 and 1. What are the correct roots of Mr. Jacobson's parabola?

Megan's roots imply that $C = 6(-4) = -24$, and Greg's roots imply that $B = -(-6 + 1) = 5$, so that the original equation must have been $r^2 + 5r - 24 = 0$, which factors to $(r + 8)(r - 3) = 0$ and thus has roots of -8 and 3.

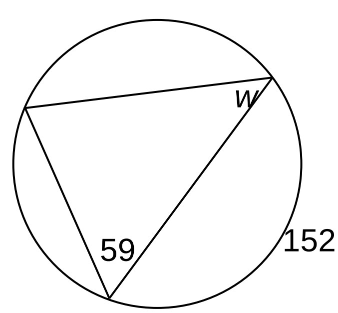
21. In Tunnel Hill, elves tend giant ants as livestock. If there are a total of 100 legs and 20 heads, how many elves are there?

If it were all ants, there would be $6 \times 20 = 120$ legs, but instead we have 20 fewer than that. Each time we trade an ant for an elf, we lose $6 - 2 = 4$ legs, so we need to do that $20 \div 4 = 5$ times, meaning we have 5 elves.

22. If $g(h) = \frac{(h+4)^2}{h-3}$, evaluate $g(10)$.

$g(10) = \frac{(10+4)^2}{10-3} = \frac{14^2}{7} = 14 \times 2 = 28$

23. In the figure to the right with a circle circumscribed around a triangle, an arc measure and two of the triangles angles are given in degrees. What is the value of w?



The unmarked angle of the triangle is $\frac{152}{2} = 76$, so that $w = 180 - 76 - 59 = 104 - 59 = 45$.

24. How many regular polygons can tessellate a plane all by themselves?

An equilateral triangle can, a square can, a pentagon cannot (because its angles are $108°$, which doesn't go into $360°$ evenly), a hexagon can, and then nothing else can, for an answer of 3.

25. A rectangle measures 8 m by 12 m. What is the area, in square meters, of all of the points that lie within one meter of at least one point on the rectangle?

The interior area that is within a meter of the rectangle is everything outside a 6 by 10 rectangle. The exterior area is almost everything inside a 10 by 14 rectangle, except that the four corners should each be quarter-circles instead of squares. Thus, our answer is $10 \times 14 - 6 \times 10 - 4 + \pi = 140 - 60 - 4 + \pi = 76 + \pi$.

26. How many diagonals can be drawn in a convex 15-gon?

You can draw 12 from the first vertex you pick, 12 from the next one over, then 11, 10, 9, … alternatively, you can use the formula $\frac{n(n-3)}{2} = \frac{15\times12}{2} = 15 \times 6 = 90$.

27. A right triangle has a hypotenuse measuring 9 m and a leg measuring 5 m. What is the length, in meters, of the other leg?

The Pythagorean Theorem gives $\sqrt{9^2 - 5^2} = \sqrt{81 - 25} = \sqrt{56} = 2\sqrt{14}$.

28. A triangle with angles measuring $30°$, $60°$, and $90°$ has a longest side measuring 18 m. What is the area, in square meters, of the triangle?

The shortest side (a leg) is half the longest (the hypotenuse), so is 9. The longer leg is $\sqrt{3}$ times the shorter, so $9\sqrt{3}$, for an area of $\frac{1}{2} \times 9 \times 9\sqrt{3} = \frac{81\sqrt{3}}{2}$.

29. A triangle with angles measuring $45°$, $45°$, and $90°$ has a longest side measuring 18 m. What is the length, in meters, of the shortest side of the triangle?

The hypotenuse is $\sqrt{2}$ times a leg, so each leg is $\frac{18}{\sqrt{2}} = \frac{18\sqrt{2}}{2} = 9\sqrt{2}$.

30. When the three medians of a triangle are drawn, they will always meet at a single point. What is a name for this point?

A median bisects the far side, so if you think about little line segments parallel to that side, each of them would be bisected by the median and thus balance on a pencil laid there. Because the point is where all three medians meet, the entire triangle would balance on the tip of a pencil placed there. This point can be called the center of balance, the center of mass, or the centroid (any of these answers would be acceptable).

31. Two sides of a triangle measure 41 m and 56 m. When the third side of the triangle is measured in meters, what is the smallest possible integer value of that measurement?

Any two sides of a triangle must add up to be more than the third side, which is called the Triangle Inequality. In this case, $41 + c > 56$ means that $c > 15$, for an answer of 16.

32. What is the area, in square meters, of a hexagon with a perimeter of 48 m?

A perimeter of 48 means each side is $\frac{48}{6} = 8$. The hexagon can be broken into 6 equilateral triangles, for an area of $6 \times \frac{8^2\sqrt{3}}{4} = 6 \times 4^2\sqrt{3} = 96\sqrt{3}$.

33. What is the circumference, in meters, of a circle with an area of 49π square meters?

An area of 49π implies a radius of $\sqrt{49} = 7$, a diameter of $2 \times 7 = 14$, and a circumference of 14π.

34. What is the volume, in cubic meters, of a right circular cone with a base radius of 12 m and a height of 5 m?

$V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h = \frac{\pi}{3} \times 12^2 \times 5 = \pi \times 4 \times 12 \times 5 = 20 \times 12\pi = 240\pi$

35. The supplement of the complement of an angle measures $119°$. What was the measure, in degrees, of the original angle?

If the supplement is 119, the intermediate angle is $180 - 119 = 61$. If 61 is the complement of the original angle, that angle is $90 - 61 = 29$.

36. Regular 24-gon Z has vertices labeled A-X in clockwise order. Evaluate $m\angle ADV$, in degrees.

Draw a circle around the 24-gon; each arc should measure $\frac{360}{24} = 15°$. $\angle ADV$ has its vertex on the circle, and subtends three of these arcs, for a measure of $\frac{3\times 15}{2} = \frac{45}{2}$.

37. What value(s) of q satisfy $4^{2q} = 48 \times 4^q - 512$?

This is a "hidden quadratic," as 4^{2q} is the square of 4^q. Bringing everything onto one side gives $4^{2q} - 48 \times 4^q + 512 = 0$, which factors to $(4^q - 16)(4^q - 32) = 0$, so $4^q = 16$ or $4^q = 32$. You could turn these into powers of 2, but $q = 2$ or $q = \frac{5}{2}$.

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38. Use the table to the right to evaluate $\boldsymbol{f\left(g\left(h^{-1}\left(f\left(g^{-1}(1)\right)\right)\right)\right)}$.

	$z = 1$	$z = 2$	$z = 3$	$z = 4$	$z = 5$	$z = 6$
$f(z)$	1	5	2	6	4	3
$g(z)$	4	2	1	5	3	6
$h(z)$	6	4	3	2	5	1

$f\left(g\left(h^{-1}\left(f\left(g^{-1}(1)\right)\right)\right)\right) =$
$f\left(g\left(h^{-1}(f(3))\right)\right) =$
$f\left(g\left(h^{-1}(2)\right)\right) = f\left(g(4)\right) = f(5) = 4$

39. The function $\boldsymbol{j(k) = \frac{2k^2}{k^2+1}}$ **has a domain and a range that are subsets of the real numbers. Express the range in interval notation.**

You can think about the function as written, but it may be easier to express it as $j(k) = 2 - \frac{2}{k^2+1}$. Obviously, for large k (positive or negative), this approaches 2, but for small k (0), it achieves 0, so that the range is $[0,2)$.

40. What is the product of the reciprocals of the roots of $\boldsymbol{3m^3 - 2m^2 + m - 4 = 0}$**?**

There are three roots, r, s, and t, and the product of their reciprocals is $\frac{1}{rst}$, which is just the reciprocal of their product. The product of the roots of a polynomial is $(-1)^n \times \frac{z}{a}$, which in this case is $(-1)^3 \times \frac{-4}{3} = \frac{4}{3}$, the reciprocal of which is $\frac{3}{4}$.

41. What are the coordinates, in the form $\boldsymbol{(x, y)}$**, of the center of the conic with equation** $\boldsymbol{9x^2 - 2y^2 - 36x - 24y = 100}$**?**

Completing the squares would lead to $9(x-2)^2 - 2(y+6)^2 = \cdots$, for a center of $(2, -6)$.

42. How many positive integers are factors of 540?

$540 = 2^2 \times 3^3 \times 5^1$, so any factor can have zero to two 2's (three choices), zero to three 3's (four choices), and zero or one 5 (two choices), for a total of $3 \times 4 \times 2 = 24$ factors.

43. What is the product of the smallest and largest three-digit palindromes?

The smallest three-digit palindrome is 101, and the largest is 999, for a product of 100899.

44. List which of the following numbers are divisible by 12:
53142 97568 35962 14796 98632 18514 82608

To be divisible by 12, it must be divisible by 4 and 3. To be divisible by 4, the last two digits must form a two-digit number that is divisible by 4. 68, 96, 32, and 08 are divisible by 4, so 97568, 14796, 98632, and 82608 are in the running. To be divisible by 3, the digits must add up to a multiple of 3, so 97568 and 98632 don't make the cut, for an answer of 14796 & 82608.

45. Event R has a probability of $\frac{1}{4}$ and Event S has a probability of $\frac{2}{3}$. If these events are mutually exclusive, what is the probability of $R \cap S$?

Mutually exclusive means there is no way both events can occur, so $p(R \cap S) = 0$.

46. In how many distinguishable ways can the letters in the word "TALLAHASSEE" be arranged?

$\frac{11!}{3!\times 2!\times 2!\times 2!} = 11 \times 10 \times 9 \times 7 \times 6 \times 5 \times 4 = 990 \times 840 = 831{,}600$

47. What is the sum of the perfect squares less than 1000?

$31^2 = 961$ and $32^2 = 1024$, so we want the sum of the first 31 perfect squares. We can use the formula $\frac{n(n+1)(2n+1)}{6} = \frac{31\times 32\times 63}{6} = 31 \times 16 \times 21 = 651 \times 16 = 10{,}416$.

48. What is the 1234th term of an arithmetic sequence with a first term of 56 and a common difference of 78?

The 1234th term will be 1233 differences from the first term, for an answer of $56 + 1233 \times 78 = 96{,}230$.

49. What is the smallest possible sum of an infinite geometric sequence with a third term of 18 and a fifth term of 2?

The common ratio is $\sqrt{\frac{2}{18}} = \sqrt{\frac{1}{9}} = \pm\frac{1}{3}$, so that the first term is $18 \times 3^2 = 18 \times 9 = 162$ and the smallest possible sum is $\frac{162}{1-\left(-\frac{1}{3}\right)} = \frac{162}{\frac{4}{3}} = \frac{162\times 3}{4} = \frac{81\times 3}{2} = \frac{243}{2}$.

50. Evaluate: $\boldsymbol{< 1, 3, -5 > \cdot < 4, -2, -1 >}$

$< 1,3,-5 > \cdot < 4,-2,-1 > = 1 \times 4 + 3(-2) + (-5)(-1) = 4 - 6 + 5 = 3$

51. In a five-element data set of integer test scores from 0 to 100 inclusive, the mean is 70 and the median is 60. What is the smallest possible value of the range?

In ascending order, the elements are a, b, 60, c, d. We know that $a+b+c+d+60=5\times 70=350$, so $a+b+c+d=290$. If one of c or d gets larger, one of a or b will need to get smaller to keep the total constant, but to have a small range we'd like the reverse to happen, so we'll have a and b be as high as possible (60) and c and d be as low as possible in concert with that. $60+60+c+c=290$ becomes $120+2c=290$, then $2c=170$, so that $c=85$. This means our desired data set is $60, 60, 60, 85, 85$, and that our answer is $85-60=25$.

52. What is the product of the median and range of the data set $\{7, 2, 9, -3, 5\}$?

In ascending order, the elements are -3, 2, 5, 7, 9, so that the median is 5 and the range is $9-(-3)=12$, for a product of 60.

53. Set B is the set of positive even numbers less than 20, and Set L is the set of positive multiples of 4 less than 10. How many subsets of set B are supersets of Set L?

L is {4,8}, while B is {2,4,6,8,10,12,14,16,18}. The desired sub/superset must contain 4 and 8, then can have or not have each of 2, 6, 10, 12, 14, 16, and 18 (two choices each). Thus, there are $2^7=128$ possible sub/supersets.

54. What is the measure of the angles at the corners of the gigantic rectangular geologic feature recently discovered on the moon?

Rapidly cooling lava tends to crack at $120°$ angles, which for small features results in hexagons. However, for a very large feature such as that recently identified on the moon, it produced something that appears rectangular because of the curvature of the sphere.

55. Convert $\frac{\pi}{7}$ into degrees, minutes, and seconds to the nearest second in the form $d°m's''$.

$180°\div 7=25r5$, so our answer will include $25°$ and we'll need to figure out what $\frac{5}{7}$ of a degree is. In minutes, that would be $\frac{5\times 60}{7}=300\div 7=42r6$, so now we have $25°42'$ and need to figure out the seconds. $\frac{6}{7}\times 60=360\div 7=51r3$, for an answer of $25°42'51''$.

56. If $\sin k=\frac{5}{13}$, what is the largest possible value of $\sin\frac{k}{2}$?

You can memorize half-angle formulae, but I've never bothered. I know they're based on the cosine double-angle formula, so I say $\cos k=1-2\sin^2\frac{k}{2}$, then $2\sin^2\frac{k}{2}=1-\cos k$, then $\sin\frac{k}{2}=\pm\sqrt{\frac{1-\cos k}{2}}$. If $\sin k=\frac{5}{13}$, $\cos k=\pm\sqrt{1-\left(\frac{5}{13}\right)^2}=\pm\sqrt{\frac{169-25}{169}}=\pm\sqrt{\frac{144}{169}}=\pm\frac{12}{13}$. To get the largest possible value of $\sin\frac{k}{2}$, we'll use the negative value of $\cos k$ and the positive value of $\sin\frac{k}{2}$ to get $\sin\frac{k}{2}=\sqrt{\frac{1+\frac{12}{13}}{2}}=\sqrt{\frac{25}{26}}=\frac{\sqrt{650}}{26}$.

57. Express the cylindrical coordinates $\left(8, \frac{5\pi}{3}, 2\right)$ in rectangular form (x, y, z).

Cylindrical coordinates are typically in the order (r, θ, z), so $z = 2$ is easy, and $x = 8\cos\left(\frac{5\pi}{3}\right) = 8 \times \frac{1}{2} = 4$ and $y = 8\sin\left(\frac{5\pi}{3}\right) = 8 \times \left(-\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$, for an answer of $\left(4, -4\sqrt{3}, 2\right)$.

58. What is the area of the region bounded by $2 < x < 4$ and $0 < y < x^3$?

$$\int_2^4 x^3 dx = \frac{1}{4}x^4\Big|_2^4 = \frac{1}{4}(4^4 - 2^4) = \frac{1}{4}(256 - 16) = \frac{1}{4} \times 240 = 60$$

59. Evaluate as a mixed number: $5\frac{11}{135} \div 1\frac{37}{75}$

$$5\frac{11}{135} \div 1\frac{37}{75} = \frac{686}{135} \div \frac{112}{75} = \frac{686}{135} \times \frac{75}{112} = \frac{343}{27} \times \frac{15}{56} = \frac{49}{9} \times \frac{5}{8} = \frac{245}{72} = 3\frac{29}{72}$$

60. Evaluate: $65^3 - 35^3$

$65^3 - 35^3 = (65 - 35)(65^2 + 65 \times 35 + 35^2) = 30((65 + 35)^2 - 65 \times 35) = 30\left(100^2 - (50^2 - 15^2)\right) = 30(10{,}000 - 2275) = 30 \times 7725 = 231{,}750$

61. Simplify by multiplying and combining like terms: $(v - 1)v(2v + 8)(3 - v)$

$(v - 1)v(2v + 8)(3 - v) = v(2v^2 + 6v - 8)(3 - v) = v(-2v^3 + 26v - 24) = -2v^4 + 26v^2 - 24v$

62. Express the solution to the system of equations $u + 2t + 3s = -6$, $2u - 4t + s = 9$, and $3u + 2t - 2s = 1$ as an ordered triple in the form (u, t, s).

Subtracting twice the first equation from the second gives $-8t - 5s = 21$, while subtracting three times the first equation from the third gives $-4t - 11s = 19$. Subtracting two of this final equation from the one before gives $17s = -17$, so that $s = -1$. Substituting this into the previous equation gives $-4t + 11 = 19$, then $-4t = 8$, so that $t = -2$. Finally, substituting both of these into the first equation gives $u - 4 - 3 = -6$, so that $u = 1$, for an answer of $(1, -2, -1)$.

63. Consider the graphs of Parabola H $(y - k = a(x - h)^2)$ and Parabola J $(y - k = a(x - j)^2)$, where a, h, j, and k are constants and $h > j$. List all of the following phrases that would be accurate in the sentence "Parabola H is __________ Parabola J."

above	**to the left of**	**narrower than**	**taller than**
below	**to the right of**	**wider than**	**shorter than**

These parabola are the same except for h and j, which influence the x-variable. In H, $x = h$ is the axis of symmetry, while in J, $x = j$ is the axis of symmetry. Because $h > j$, the axis of symmetry of H is to the right of that of J, making the answer "to the right of" (we'll accept "right"), but none of the other options.

64. All the families in the cul-de-sac contribute the same amount of money towards the purchase of a riding lawn mower that anyone can use to mow their lawn. If there had been one fewer family, each family would have had to pay $274 more than they actually did. If there had been three more families, each family would have paid $685 less than they actually did. How many dollars did the riding mower cost?

If F is the number of families and P is the amount they each paid, we can write $FP = (F-1)(P+274) = (F+3)(P-685)$, and can break this into two different equations: $FP = (F-1)(P+274)$ and $FP = (F+3)(P-685)$. The first becomes $FP = FP - P + 274F - 274$, then $0 = -P + 274F - 274$, and finally $274 = -P + 274F$. Similarly, the second equation becomes $2055 = 3P - 685F$. Adding three of the former to the latter gives $2055 + 822 = 822F - 685F$, then $2877 = 137F$, so that $F = 21$, leading to $P = 20 \times 274 = 5480$, for an answer of $21 \times 5480 = 115{,}080$.

65. I've got 37 coins in my pocket worth a total of $6.00. If I have at least one penny, nickel, dime, and quarter, and no other types of coins, what is the largest number of nickels I could have?

If I have at least one penny, I must have at least 5. Along with the required nickel, dime and quarter, that means I've already used 8 of my 37 coins and $0.45 of my $6.00, so there are 29 remaining coins worth $5.55. This means the average value of each coin needs to be around $0.20, so we need to go heavy on quarters. If we restrict ourselves to quarters and nickels, that will give us the most nickels. 20 quarters & 9 nickels would be $5.45, which is very close to what we want. 21 quarters and 8 nickels would be $5.65, so we're going to have to use some dimes to make it work perfectly. 20 quarters, 1 dime, and 8 nickels would be $5.50, and 20 quarters, 2 dimes, and 7 nickels would be $5.55. These 7 nickels and the one we already allocated makes 8.

66. A rectangle with an aspect ratio of 2 is inscribed in a circle with a radius of 3. What is the area, in square meters, of the rectangle?

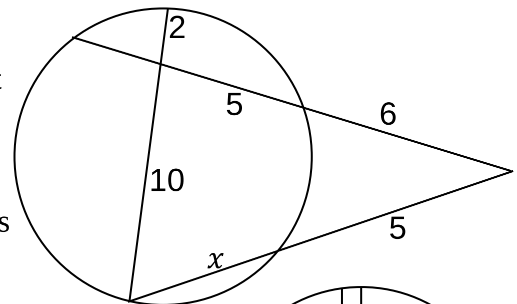
The diagonal of the rectangle must be $2 \times 3 = 6$. If the sides of the rectangle are s and $2s$, the Pythagorean Theorem gives $6^2 = 36 = s^2 + (2s)^2 = s^2 + 4s^2 = 5s^2$. The area of the rectangle will be $s \times 2s = 2s^2 = \frac{36\times2}{5} = \frac{72}{5}$.

67. When the measures of seven of the angles of a convex decagon are added together, the result is 987°. If all angles are integers when measured in degrees, what is the smallest possible number of degrees in the measure of one of the three other angles?

The interior angles of a decagon sum to $(10-2)\times 180 = 8\times 180 = 1440$, meaning the remaining three angles must add up to $1440 - 987 = 453$. None of the angles can be $180°$ or more, so the maximum measure for two of them is $179°$, making the smallest possible value of the third $453 - 2\times 179 = 453 - 358 = 95$.

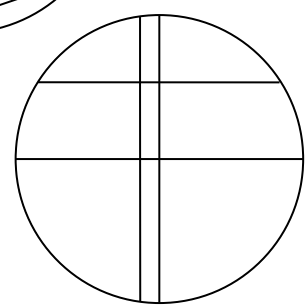
68. What is the volume, in cubic meters, of a regular octahedron with edges measuring 8 m?

If you chop an octahedron into eight pieces, each of them is a right triangular pyramid that is one-sixth of a cube. The edges of the cube will be $\frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$, so the volume of the octahedron will be $8\times\frac{1}{6}\times\left(4\sqrt{2}\right)^3 = \frac{4}{3}\times 128\sqrt{2} = \frac{512\sqrt{2}}{3}$.



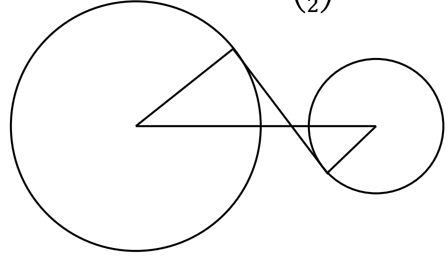
69. In the figure to the right, several line segments intersect a circle and one another, and all segment lengths are given in meters. What is the value of x?

The two chords imply that the unmeasured segment is $\frac{2\times 10}{5} = 4$. The two secants then imply that $6\times(6+5+4) = 5(5+x)$, which becomes $6\times 15 = 5(5+x)$, then $6\times 3 = 5+x$, then $18 = 5+x$, finally giving $x = 18-5 = 13$.



70. Two chords in a circle are perpendicular, and divide each other into lengths of 2 m, 3 m, 4 m, and t m (in no particular order). What is the largest possible area, in square meters, of the circle?

As in the previous problem, when two chords intersect, the product of their segment lengths should be equal. If the 2 & 3 are part of the same chord, then the 4 must be paired with $\frac{2\times 3}{4} = \frac{6}{4} = \frac{3}{2} = 1.5$, for a diameter on the order of $4 + 1.5 = 5.5$. If the 2 & 4 are part of the same chord, then the 3 must be paired with $\frac{2\times 4}{3} = \frac{8}{3} \approx 2.67$, for a diameter on the order of $2+4=6$. If the 3 & 4 are part of the same chord, then the 2 must be paired with $\frac{3\times 4}{2} = \frac{12}{2} = 6$, for a diameter on the order of $2+6=8$, so this case will produce the largest area. The figure above has these chords drawn in their circle, as well as the diameters parallel to them. One of these diameters is .5 from the 2-6 chord, while the other is 2 from the 3-4 chord. The Pythagorean Theorem then shows the square of the radius to be $4^2 + \left(\frac{1}{2}\right)^2 = 2^2 + \left(\frac{7}{2}\right)^2 = \frac{65}{4}$, for an answer of $\frac{65\pi}{4}$.



71. Two circles with respective radii of 8 m and 16 m have their centers 30 m apart. What is the length, in meters, of one of their common internal tangents?

In the figure above, the tangent, two radii, and the segment between the centers are drawn, making two similar right triangles. Because of similarity, one of them gets $\frac{8}{8+16} = \frac{8}{24} = \frac{1}{3}$ of the 30, which is $\frac{1}{3} \times 30 = 10$, and the other gets the remaining 20. The Pythagorean Theorem makes the shorter portion of the tangent $\sqrt{10^2 - 8^2} = \sqrt{100 - 64} = \sqrt{36} = 6$, so that the other portion is $2 \times 6 = 12$ and the whole thing is $6 + 12 = 18$.

72. What is the largest real value of *r* for which there is a real value of *s* satisfying $3r^2 + 2s^2 + rs + 3r + 2s = 6$?

For any particular value of r, the value(s) of s are $\frac{-(r+2)\pm\sqrt{(r+2)^2-4\times2\times(3r^2+3r-6)}}{2\times2}$. There will be two values of s unless the quadratic is zero, in which case there will be just one value of s, and if the quadratic is negative then there will be no values of s. Thus, we want the largest value of r that will make the quadratic zero. $(r+2)^2 - 4 \times 2 \times (3r^2 + 3r - 6) = 0$ becomes $r^2 + 4r + 4 - 24r^2 - 24r + 48 = 0$, then $-23r^2 - 20r + 52 = 0$, with roots of $\frac{20\pm\sqrt{(-20)^2-4(-23)\times52}}{2(-23)} = \frac{20\pm\sqrt{400+208\times23}}{-46} = \frac{-20\pm2\sqrt{100+52\times23}}{46} = \frac{-20\pm4\sqrt{25+13\times23}}{46} = \frac{-10\pm2\sqrt{324}}{23} = \frac{-10\pm2\times18}{23} = \frac{-10\pm36}{23}$. We want the larger root, which is $\frac{-10+36}{23} = \frac{26}{23}$.

73. At how many points do at least two of the graphs of $x^2 + \frac{y^2}{4} = 100$, $(x-1)^2 - \frac{(y+2)^2}{4} = 25$, and $\frac{(x+3)^2}{9} + (y-4)^2 = 49$ intersect?

The first is a 20 by 40 ellipse centered at the origin, the third is a 42 by 14 ellipse centered at $(-3,4)$, and the second is a hyperbola centered at $(1,-2)$. Each of these intersects each of the others at four points, for an answer of 12.

74. If $\log_2 3 = n$ and $\log_3 5 = p$, evaluate $\log_{10} 27$.

$\log_{10} 27 = 3\log_{10} 3 = \frac{3}{\log_3 10} = \frac{3}{\log_3 2+\log_3 5} = \frac{3}{\frac{1}{n}+p} = \frac{3n}{1+pn}$

75. Express the base nine number 248_9 as a base three number.

Each digit in base nine is two digits in base three because $9 = 3^2$, so $248_9 = 21122_3$.

76. If $q \equiv 3 \; mod \; 6$ and $q \equiv 5 \; mod \; 8$, what is the smallest possible value of $q > 1000$?

The least common multiple of 6 and 8 is 24, so a number in each 24 will satisfy the constraints. Considering 5, 13, and 21, the last satisfies both constraints, so we're looking for numbers of the form $21 \; mod \; 24$. $1000 \equiv 16 \; mod \; 24$, so the smallest number greater than 1000 that satisfies both constraints is 1005.

77. Unit squares are used to create a 12 by 18 rectangle, and one of its diagonals is drawn. How many unit squares does this diagonal pass through (merely touching a corner does not count)?

The diagonal passes through the corners of six smaller 2 by 3 rectangles, and in each of them it intersects four squares, for an answer of $6 \times 4 = 24$.

78. A trusted friend rolls two standard dice behind a screen and tells you she didn't get any odd numbers. What is the probability that she rolled a sum of six?

There are only three even numbers on each die, so there are only $3 \times 3 = 9$ possible rolls. Of these, only 2&4 or 4&2 (two cases) produce a sum of 6, for a probability of $\frac{2}{9}$.

79. If two points are chosen on a meter stick, what is the probability they are within 20 cm of one another?

Consider a square graph like the x-y plane of all of the values from 0 to 100 that the two points might have on the yardstick. To be within 20 cm of one another, we'd like $x + 20 > y > x - 20$, which is a diagonal stripe through our square. The easier area to calculate is the triangles that *aren't* in our good zone, so that the probability becomes $\frac{100^2-80^2}{100^2} = \frac{10^2-8^2}{10^2} = \frac{100-64}{100} = \frac{36}{100} = \frac{9}{25}$.

80. Two people play a game in which they take turns rolling a fair six-sided dice. They take turns until somebody wins by rolling a number equal to or greater than the number just rolled by the other player. According to these rules, the first player cannot win on her first turn. What is the probability that the second player wins the game eventually?

The second player can win on Turn 2, Turn 4, or Turn 6, but no later, as eventually someone wins. To win on turn 2, the rolls must go 1x (6 ways), 2x (5 ways), 3x (4 ways), 4x (3 ways), 5x (2 ways), or 66 (one way), for a total of 21 ways out of 36. To win on Turn 4, the rolls must go yy1x ($5c2 \times 6 = 10 \times 6 = 60$ ways), yy2x ($4c2 \times 5 = 6 \times 5 = 30$ ways), yy3x ($3c2 \times 4 = 3 \times 4 = 12$ ways), or yy4x ($2c2 \times 3 = 1 \times 3 = 3$ ways), for a total of 105 ways out of 1296 ways. To win on Turn 6, the rolls must go zzzz1x ($5c4 \times 6 = 5 \times 6 = 30$ ways) or zzzz2x ($4c4 \times 5 = 1 \times 5 = 5$ ways), for a total of 35 ways out of 46656 ways. These can be added to get $\frac{31031}{46656}$.

81. Five distinguishable keys, each with a flat side and a bumpy side, are placed on a standard key ring. How many distinguishable arrangements are possible?

There are $4! = 24$ ways to arrange the keys around the ring, just like around a table. The ring can be flipped over, so the orientation of the first key won't matter, but the orientation of the others relative to it will, so we must multiply by $2^4 = 16$, for an answer of $16 \times 24 = 20^2 - 4^2 = 400 - 16 = 384$.

82. As the millionth customer at the casino, you're going to win money by playing a special game! You'll flip a coin eight times; if you're first flip is heads, you'll win \$1. If your first two flips are heads, you'll win an additional \$2. If your first three flips are all heads, you'll win an additional \$6. If your first four flips are all heads, you'll win an additional \$24. If your first five flips are all heads, you'll win an additional \$120. If your first six flips are all heads, you'll win an additional \$720. If your first seven flips are all heads, you'll win an additional \$5040. Finally, if you flip either eight heads or eight tails, you'll win \$1,000,000! (For all heads, the \$1,000,000 is in addition to your earlier winnings.) What are your expected winnings, in dollars rounded to the nearest hundredth (cent)?

There is a $\frac{1}{2}$ chance you receive the \$1, a $\frac{1}{4}$ chance of \$2, etc., for an expected value of $\frac{1}{2}+\frac{2}{4}+\frac{6}{8}+\frac{24}{16}+\frac{120}{32}+\frac{720}{64}+\frac{5040}{128}+\frac{2\times 1000000}{256}=\frac{1}{2}+\frac{1}{2}+\frac{3}{4}+\frac{3}{2}+\frac{15}{4}+\frac{45}{4}+\frac{315}{8}+\frac{5^6}{2}=\frac{5}{2}+\frac{63}{4}+\frac{315}{8}+\frac{15625}{2}=\frac{15630}{2}+\frac{63}{4}+\frac{315}{8}=\frac{62520+126+315}{8}=\frac{62961}{8}\approx 7{,}870.13.$

83. At the same casino, you're allowed to draw a marble from a bag containing two red and three green marbles, but you can't look at it; you just keep it clutched in your hand. After you choose, the Pit Boss roots around in the bag (he looks) and removes two green marbles, which he shows you and then throws into the watching crowd. After this, he gives you a chance to throw your marble into the crowd (still not looking, and the crowd remains deathly silent) and select a new marble from the bag. You could also choose to keep the marble in your hand (without looking at it). If you end up with a red marble, you Win Big and get Lots of Money, otherwise you Lose and get Nothing. If you choose to throw your first marble away and select a second marble, what is the probability you end up with a red marble?

For the first marble, there is a $\frac{2}{5}$ chance it's red and $\frac{3}{5}$ chance it's blue. If it's red, then there's a red and a green once the Pit Boss gets rid of the two greens, so when you draw again the probability of a red is $\frac{1}{2}$. If the first marble was green, then there are two reds after the two greens are taken out, so when you draw again the probability of a red is 1. This makes the answer $\frac{2}{5}\times\frac{1}{2}+\frac{3}{5}\times 1=\frac{1}{5}+\frac{3}{5}=\frac{4}{5}$.

84. Evaluate: $\sum_{z=2}^{\infty}\frac{1}{z^2-z}$

$\sum_{z=2}^{\infty}\frac{1}{z^2-z}=\sum_{z=2}^{\infty}\left(\frac{1}{z-1}-\frac{1}{z}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$ The negative part of each term is canceled by the positive part of the next term, so that only the 1 remains in the end.

85. What is the missing term of the sequence 1, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{2}$, $\frac{24}{5}$, 20, $\frac{720}{7}$, ___, 4480, 36288, …

We seem to be dividing by n in each term, because of the 2, 3, 5, and 7. The numerators of these terms are all factorials, specifically $(n-1)!$ This would make the missing term $\frac{7!}{8}=7\times 6\times 5\times 3=630$.

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86. What is the ninth term of a recursive sequence with first term $t_1 = 1$ and nth term $t_n = 2t_{n-1} + 3$?

The terms are 1, $2 \times 1 + 3 = 5$, $2 \times 5 + 3 = 13$, $2 \times 13 + 3 = 29$, $2 \times 29 + 3 = 61$, etc. Hey, these are all near powers of 2! Specifically, they're all three less than a power of 2, so that our answer will be $2^{10} - 3 = 1024 - 3 = 1{,}021$.

87. What is the sum of the positive four-digit even numbers?

There are 4999 evens up to 9998 which sum to 4999×5000, but we don't want the ones up to 998 that add up to 499×500, so our answer is $4999 \times 5000 - 499 \times 500 = 25000000 - 5000 - (250000 - 500) = 25000500 - 255000 = 24{,}745{,}500$.

88. The matrix $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ has an infinite number of eigenvectors in the form $\begin{bmatrix} x \\ y \end{bmatrix}$. List the two which are linearly independent of one another, have a positive x-component, and have components which are relatively prime integers.

The eigenvalues can be found by seeking a zero determinant of $\begin{bmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix}$, so that $(3-\lambda)(2-\lambda) - 1 \times 2 = 6 - 5\lambda + \lambda^2 - 2 = \lambda^2 - 5\lambda + 4 = 0$, which factors to $(\lambda - 4)(\lambda - 1) = 0$ with roots of 4 and 1. This makes the "eigenmatrices" $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$, which have simplified eigenvectors of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

89. In the data set of integers $\{8, u, -2, v, 4, w, 1, 4, 2\}$, the single mode is greater than the median, which is greater than the mean. What is the largest possible value of $u + v + w$?

The known terms, in ascending order, are -2, 1, 2, 4, 4, 8, with u, v, and w being in there somewhere. Depending on where u, v, and w are, the median could be anything from 1 (all low) to 4 (all high). Thus, the mode must be greater than (not equal to) 1, and the mean must be less than (not equal to) 4. We want the largest value of $u + v + w$, which means we want the largest value of the mean, which means we want the mean to be just barely below the median, which we'd also like high, so we'd like the mode to be as high as possible, too.

If the mode remains 4, we can only have one more value that is 4 or more to keep the median below the mode of 4. This means we could have a median of 3, and any large and small value that would bring the mean to $\frac{26}{9}$, the largest value it could have below 3. This mean would require $u + v + w = 26 - 17 = 9$.

If the mode is larger than 4, it must either be $u = v = w$, or it must be 8. In the former case, the median is 4, so the mean can be at most $\frac{35}{9}$, so that $u + v + w \leq 35 - 17 = 18$, so u, v, and w could all be 6 for an answer of 18. If the mode is 8, again the median will be 4, so again the mean can be at most $\frac{35}{9}$, so again $u + v + w \leq 35 - 17 = 18$, which would require 8, 8, and 2, which is the same answer of 18.

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90. Set V is the set of people who like Vanilla, and contains 31 elements. Set C is the set of people who like Chocolate, and contains 41 elements. If the set of all people surveyed contains 61 people, what is the largest possible number of elements in the set $V' \cup C'$?

$V' \cup C'$ means everything but $V \cap C$, so we want the smallest number of people who could like both Vanilla and Chocolate, then we'll need to subtract that from 61. If the number who like both is b, then $b + (41 - b) + (31 - b) \leq 61$, which becomes $72 - b \leq 61$, giving $b \geq 11$. We want the smallest possible value of b, which is 11, so our answer is $61 - 11 = 50$.

91. In the cryptarithm $STE + MATH = STEM$, each instance of a particular letter represents the same digit (0-9) and no two different letters represent the same digit (e.g. if an A is a 1, then all A's are 1's and B's are not 1's). What is the largest possible value of the five-digit number STEAM?

We want a high value of STEAM; can S be 9? If so, M must be 8, which currently looks plausible. Can T be 7? No, because then E would have to be 4 and H would also have to be 4, or E would have to be 5 and $E + H = 8$ would have to carry which it cannot. Can T be 6? No, because then E would have to be 2 and H would have to be 6, or E would have to be 3 and $E + H = 8$ would have to carry which it cannot. Can T be 5? Also no, for similar reasons. Can T be 4? Also no. Can T be 3? Yes! It T is 3, E is 6, H is 2, and A is 4, making $STEAM = 93{,}648$.

92. You invite eight friends (A-H) over to watch a movie, but they all decline. You press them on the issue, and get the following responses in order, with each of them having heard all of the earlier responses:

A: I'm going on a date with D tonight.
B: That's a lie!
C: G & I are going on a date tonight.
D: I'm going on a date with E tonight.
E: I'm planning to wash my car tonight.
F: No one has told the truth yet!
G: I'm going on a date with C tonight.
H: At least three people have told the truth so far.

"I" refers to the speaker, rather than to another friend. Each friend is only doing one thing tonight, each date involves exactly two people, and washing a car is NOT a date. Of the eight statements, the smallest possible number are lies. List the letters of the people who must have told you the truth.

A & B must have one lie and one truth, which means F must be lying. C & G can both be telling the truth. D & E must have one lie and one truth. H can be telling the truth, too. This means that the smallest possible number of lies is 3: One of A & B, F, and one of D&E. In these scenarios, the only people who are certain to be telling the truth are C, G, and H.

93. When ten friends sit in a row at the movies, Annie sits within two seats of Billy (at most one person between them), Cindy does NOT sit within three seats of Darian, Elaine sits next to Francis, Ginny does NOT sit next to Harry, and Indiana sits within two seats of Justine. In addition, Annie sits somewhere to the left of Cindy, Billy sits somewhere to the right of Elaine, Darian sits immediately to the left of Francis, Ginny sits immediately to the right of Justine, and Harry and Indiana do NOT sit within two seats of one another. Finally, Cindy sits on an end, Harry is one of the two centermost friends, Billy sits somewhere to the left of Ginny, and Annie sits next to Justine. Write the first letters of each friend in the order they sit from left to right. E.g. AJBICHDGEF.

For problems like this, taking a lot of well-organized notes seems to work best. I wrote Ax-B, Cxxx+D, EF, NOT GH, Ix-J. Because I was working with pencil & paper, I actually put the +'s and –'s above the X's, and crossed out my GH. In another area, I wrote A-C, E-B, DF, JG, and then Hxx+I in the first area. Finally, I wrote ENDC and AJ in the first area and ?H? and B-G in the second area. Next I started connecting these notes to one another. We can build DFE-B-CX and AJG in the second area, then because A must be within two of Billy, we can combine them to be DFE-Bx-AJG-CX. H must be 5th or 6th; the only way to have H to the left of B is to have DFEIHBAJGC, which requires I next to H, which is not allowed. Thus, H must be to the right of B, giving DFE-BHAJG-CX. I must be far from H and close to J, for an answer of DFEBHAJGIC.

94. Evaluate: $\frac{1}{2+\frac{3}{4+\frac{1}{2+\frac{3}{4+\cdots}}}}$

Let's call the answer f. We can then write $f=\frac{1}{2+\frac{3}{4+f}}=\frac{1}{\frac{11+2f}{4+f}}=\frac{4+f}{11+2f}$, which becomes $11f+2f^2=4+f$, then $2f^2+10f-4=0$, then $f^2+5f-2=0$. The Quadratic Formula gives $f=\frac{-5\pm\sqrt{5^2-4(-2)}}{2}=\frac{-5\pm\sqrt{25+8}}{2}=\frac{-5\pm\sqrt{33}}{2}$, but we need a positive answer, which is $\frac{-5+\sqrt{33}}{2}$.

95. An ant is on an edge of a cube two centimeters from a vertex, and wishes to reach a point on an edge two centimeters from the opposite vertex. What is the minimum number of centimeters the ant must walk if the cube's edges are nine centimeters long?

If we unfold two adjacent sides into a planar figure, the opposite vertices of the cube are opposite vertices of this rectangle. The diagonal is roughly the distance the ant must travel, and we can reduce this distance best by having the 2 cm offsets be in the longer direction, so that the ant must travel the diagonal of a rectangle that is 9 cm wide and $2\times 9-2\times 2=2\times 7=14$ tall, for a distance of $\sqrt{9^2+14^2}=\sqrt{81+196}=\sqrt{277}$.

96. In how many points do the graphs of $y=\log_{10}x$ and $y=\sin(\pi x)$ intersect?

The sine graph alternates between -1 and 1 with period $\frac{2\pi}{\pi} = 2$. The logarithmic graph starts at $x = 0$ at $-\infty$, races up through $(1{,}0)$, and heads to the right, crossing through $(10{,}1)$, after which it's too high for us to care about. The sine will have five periods by then; in the first period, there is a single intersection at $(1{,}0)$, after which there are two intersections per period, for an answer of $1 + 2 \times 4 = 1 + 8 = 9$.

97. Simplify into a product of two of the six basic trigonometric functions:

$$\sin u \cos^2 u \tan u + \cos(u)\left(\frac{\cos u}{\sec u} + \cot^2 u\right)$$

$\sin u \cos^2 u \tan u + \cos(u)\left(\frac{\cos u}{\sec u} + \cot^2 u\right) = \sin^2 u \cos u + \cos^3 u + \frac{\cos^3 u}{\sin^2 u} = \cos u\left(\sin^2 u + \cos^2 u + \frac{\cos^2 u}{\sin^2 u}\right) = \cos u\left(1 + \frac{\cos^2 u}{\sin^2 u}\right) = \cos u\left(\frac{\sin^2 u + \cos^2 u}{\sin^2 u}\right) = \cos u\left(\frac{1}{\sin^2 u}\right) = \frac{\cos u}{\sin^2 u} = \csc u \cot u$

$\sin u \cos^2 u \tan u + \cos(u)\left(\frac{\cos u}{\sec u} + \cot^2 u\right) = \sin^2 u \cos u + \cos^3 u + \frac{\cos^3 u}{\sin^2 u} = \cos u\left(\sin^2 u + \cos^2 u + \frac{\cos^2 u}{\sin^2 u}\right) = \cos u\left(1 + \frac{\cos^2 u}{\sin^2 u}\right) = \cos u\left(\frac{\sin^2 u + \cos^2 u}{\sin^2 u}\right) = \cos u\left(\frac{1}{\sin^2 u}\right) = \frac{\cos u}{\sin^2 u} = \csc u \cot u$

98. An athlete throws a 10 pound shot put at an angle $30°$ above the horizontal and releases it at a speed of 192 feet per second. If the shot put lands on a shelf at the same height as the shot put was released, how far is the shelf from the athlete? Use the standard whole number for the acceleration due to gravity in $\frac{ft}{\sec^2}$.

The vertical velocity is $192 \sin 30° = 192 \times \frac{1}{2} = 96$, and the horizontal velocity is $96\sqrt{3}$. The acceleration due to gravity is 32 feet per second, so it will take $\frac{96}{32} = 3$ seconds for the shot to stop rising, and another 3 seconds for it to fall back to the shelf. In six seconds it will travel $6 \times 96\sqrt{3} = 576\sqrt{3}$ feet horizontally.

99. Express $\lim_{n\to 2} \frac{\frac{1}{2} - \frac{1}{n}}{2-n}$ in terms of a function $f(x)$ and its derivative at a point (e.g. $f(x) = x^2, f'(-3)$).

This is one limit-based definition of the derivative of the function $f(x) = \frac{1}{x}$ at the point $x = 2$, for an answer of $f(x) = \frac{1}{x}, f'(2)$.

100. For the function $g(x) = x^2 + 4$ defined for $-1 < x < 3$, what is the value of c for which $g'(c)$ satisfies the Mean Value Theorem?

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The Mean Value Theorem says that for a differentiable function (no corners) there must be a place in any range where the derivative is actually equal to the average rate of change (slope) of the function over that range. Thus, we want $g'(c) = 2x = \frac{g(3)-g(-1)}{3-(-1)} = \frac{13-5}{4} = \frac{8}{4} = 2$, so $x = \frac{2}{2} = 1$.