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1. **Evaluate: $\mathbf{553896 - 222716}$**

   The standard algorithm gives 331,180.

2. **What is the remainder when 234 is divided by 15?**

   $15^2 = 225$, so the answer is $234 - 225 = 9$.

3. **Evaluate: $\mathbf{4 - (-9) - 7(-2 - (-5))(7 - 9) - 4}$**

   PEMDAS first gives $4 + 9 - 7(3)(-2) - 4 = 9 + 42 = 51$.

4. **How many complete months are in 36 years?**

   There are 12 months per year, for an answer of $12 \cdot 36 = 144 \cdot 3 = 432$.

5. **Express 94.4398637 in scientific notation rounded to five significant figures?**

   This is 9.44398637×10^1, and rounded becomes 9.4440×10^1.

6. **Evaluate: $\mathbf{{}_9C_3}$**

   ${}_9C_3 = \binom{9}{3} = \frac{9!}{3!\cdot 6!} = \frac{9\cdot 8\cdot 7}{3\cdot 2} = 3 \cdot 4 \cdot 7 = 84$

7. **Express in simplest radical form: $\mathbf{\sqrt[5]{96}}$**

   $\sqrt[5]{96} = \sqrt[5]{2^5 \cdot 3} = 2\sqrt[5]{3}$

8. **Simplify by rationalizing the denominator: $\mathbf{\frac{21}{3-\sqrt{2}}}$**

   Multiplying by a carefully chosen form of 1 gives $\frac{21}{3-\sqrt{2}} \cdot \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{21(3+\sqrt{2})}{9-2} = \frac{21(3+\sqrt{2})}{7} = 3(3 + \sqrt{2}) = 9 + 3\sqrt{2}$.

9. **Evaluate: $\mathbf{82^2 - 78^2}$**

   Instead of squaring and subtracting, we'll use the fact that $(a - b)(a + b) = a^2 - b^2$, so that $82^2 - 78^2 = (82 - 78)(82 + 78) = 4 \cdot 160 = 640$.

10. **Arrange the letters below in order of ascending value (e.g. BCDA):**
    $\boldsymbol{A = \frac{2}{6}}$, $\boldsymbol{B = 0.4}$, $\boldsymbol{C = \frac{1}{2}}$, $\boldsymbol{D = 0.71}$

    Thinking of them as decimals, they're .333, .4, .5, and .71, so they're already in order! The answer is thus ABCD.

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11. Express as a simplified fraction: $.\overline{63}$

If we let $c = .\overline{63}$, we can also write $100c = 63.\overline{63}$. When we subtract the two, we get $99c = 63$, giving $c = \frac{63}{99} = \frac{7}{11}$.

12. How many dollars are 2 quarters, 3 dimes, 9 nickels, and 4 pennies worth? For example, your answer might be 7.38.

$2 \cdot .25 + 3 \cdot .1 + 9 \cdot .05 + 4 \cdot .01 = .50 + .30 + .45 + .04 = 1.29$

13. If today is Thursday, what day of the week was it 567 days ago?

7 days ago was Thursday, as was 70, as would be 700; it's all about 7s. $567 = 7 \cdot 81$, so 567 days ago was exactly 8 weeks ago, so it was Thursday.

14. When the secret number is reduced by 35 and this result is divided by 9, the final result is 29. What is the secret number?

If the final result was 29, the intermediate result must have been $29 \cdot 9 = 261$, so that the secret number is $261 + 35 = 296$.

15. What value of c satisfies $3c + 54 = 9c - 42$?

This becomes $96 = 6c$, giving $16 = c$.

16. Simplify by combining like terms: $2f + f^2 - 6 + 9f + 9f^2 + 8 - 4f^2$

The answer will be $(1 + 9 - 4)f^2 + (2 + 9)f + (-6 + 8) = 6f^2 + 11f + 2$.

17. What value(s) of g satisfy $9g^2 = 9g + 270$?

Dividing by 9 gives $g^2 = g + 30$, then $g^2 - g - 30 = 0$, which factors to $(g - 6)(g + 5) = 0$, giving roots of 6 and -5.

18. What is the solution, in the form (k, m), of the system of equations $k + 4m = -31$ and $k + 2m = -17$?

Subtracting the second equation from the first gives $2m = -14$, so $m = -\frac{14}{2} = -7$. Substituting this gives $k + 2(-7) = -17$, which becomes $k - 14 = -17$, so that $k = -3$, for an answer of $(-3, -7)$.

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19. Zilla could paint the house in 8 hours, and Kong could do so in 12 hours. How many minutes would it take them to paint the house if they work together?

Zilla's speed is $\frac{1}{8}$ of a house per hour, and Kong's is $\frac{1}{12}$, so their combined speed is $\frac{1}{8}+\frac{1}{12}=\frac{3}{24}+\frac{2}{24}=\frac{5}{24}$ of a house each hour. Thus, it would take $\frac{1}{\frac{5}{24}}=\frac{24}{5}$ of an hour, which is $\frac{24}{5}\cdot 60=24\cdot 12=288$ minutes.

20. If 16 chickens can lay 600 eggs in 2 days, how many days would it take for 8 chickens to lay 4200 eggs?

Half the chickens would take twice as long to lay 600 eggs, which would be $2\cdot 2=4$ days. Seven times as many eggs would take seven times as long, which is $7\cdot 4=28$.

21. When 8 liters of a 60% acid solution is mixed with 2 liters of a 70% acid solution, what percentage of the resulting solution is acid?

We have four times as much 60% solution as 70% solution, so the weighted average will be closer to 60% in a ratio of 4:1, so only $\frac{1}{5}$ of the way from 60% to 70%, which is 62%.

22. In a game of Hide & Seek, Pete and Tink see one another when they are 72 feet apart, and the chase is on! If Pete chases Tink at a speed of 6 feet per second, and Tink runs away at a speed of 2 feet per second, how many seconds will it take Pete to catch Tink?

Pete can close the distance at a rate of $6-2=4$ fps, so it will take him $\frac{72}{4}=18$ seconds to catch Tink.

23. Two numbers have a sum of 97 and differ by 39. What is the smaller of the two numbers?

The smaller number will be $\frac{97-39}{2}=\frac{58}{2}=29$.

24. What is the equation, in slope-intercept ($y=mx+b$) form, of the line through the points $(-5,-7)$ and $(9,35)$?

The slope will be $m=\frac{35-(-7)}{9-(-5)}=\frac{42}{14}=3$, so the line is $y=3x+b$. Substituting $(9,35)$ gives $35=3\cdot 9+b=27+b$, so that $b=35-27=8$, for an answer of $y=3x+8$.

25. What are the coordinates, in the form (x,y), of the midpoint of the line segment connecting $(-9,6)$ and $(2,-3)$?

The midpoint will be at $\left(\frac{-9+2}{2},\frac{6+(-3)}{2}\right)=\left(-\frac{7}{2},\frac{3}{2}\right)$.

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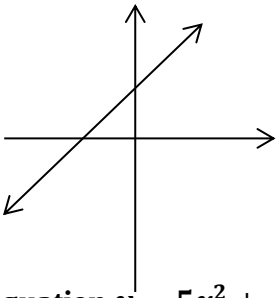
26. Ignoring scaling, which of the following equations might describe the line shown to the right?



A. $14x - 8y = 6$ **B.** $9x + 8y = 7$
C. $8x - 7y = -6$ **D.** $2x + 4y = -3$

The line has a positive y-intercept, which could be either B or C. The line also has a negative x-intercept, which limits the options to C.

27. What is the equation of the axis of symmetry of the parabola with equation $y = 5x^2 + 60x - 62$?

The axis is $x = -\frac{b}{2a} = -\frac{60}{2 \cdot 5} = -\frac{60}{10} = -6$.

28. What are the coordinates, in the form (x, y), of the vertex of the parabola with equation $y = 7x^2 + 126x - 49$?

The vertex will lie on the axis of symmetry, which is $x = -\frac{b}{2a} = -\frac{126}{2 \cdot 7} = -\frac{63}{7} = -9$, so its y-value will be $y = 7(-9)^2 + 126(-9) - 49 = -7(81) - 49 = -567 - 49 = -616$, for an answer of $(-9, -616)$.

29. When Ms. Tary puts a quadratic of the form $x^2 + Bx + C = 0$ on the board, Wily miscopies the value of B, getting roots of 2 and -9, and JT miscopies the value of C, getting roots of 5 and -8. What is the greater root of the original quadratic?

Wily had the correct C, which must have been $C = 2(-9) = -18$. JT had the correct B, which must have been $B = -\big(5 + (-8)\big) = -(-3) = 3$. The original quadratic must have been $x^2 + 3 - 18 = 0$, which factors to $(x + 6)(x - 3) = 0$, with roots of -6 and 3, for an answer of 3.

30. A ranch has only wranglers (humans) and horses. If an automated camera detects 28 heads and 108 legs, how many wranglers are on the ranch?

There are 28 creatures. If they were all horses, there would be $4 \cdot 28 = 112$ heads, which is $112 - 108 = 4$ heads too many, so we need to turn some horses in our theory into wranglers. For each horse that becomes a wrangler, we'll lose 2 legs, so we need to convert $\frac{4}{2} = 2$, which is our answer.

31. Tom is 7 times as old as Katie. 2 years ago, Katie was 7 years old. How old was Tom then?

If Katie must be $7 + 2 = 9$ now, so Tom is $7 \cdot 9 = 63$, so that he was $63 - 2 = 61$ then.

32. If $g \blacksquare h = 2gh - h^2$, evaluate $4 \blacksquare 3$.

$4 \blacksquare 3 = 2 \cdot 4 \cdot 3 - 3^2 = 24 - 9 = 15$

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33. A right triangle has a hypotenuse measuring 39 ft and one leg measuring 36 ft. What is the length, in feet, of the other leg?

This triangle is 3 times as big a triangle with a hypotenuse of 13 and a leg of 12, which you may recognize gives another leg of 5, for an answer of $3 \cdot 5 = 15$. If you did not recognize the Pythagorean triple, you could use the Pythagorean Theorem.

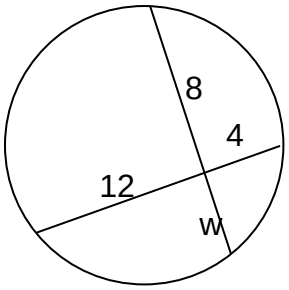
34. What is the area, in square meters, of a right triangle with legs measuring 5 m and 14 m?

A right triangle is half a rectangle (diagonally), so its area is $A = \frac{1}{2}bh = \frac{1}{2} \cdot 5 \cdot 14 = 5 \cdot 7 = 35$.

35. What is the length, in feet, of a leg of a right triangle with one angle measuring 45 degrees and a hypotenuse measuring 8 feet?

In a 45-45-90 triangle, the hypotenuse is $\sqrt{2}$ times each leg, for an answer of $\frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$.

36. The figure to the right shows a circle with two intersecting chords, with each line segment's length given in meters. What is the value of w?

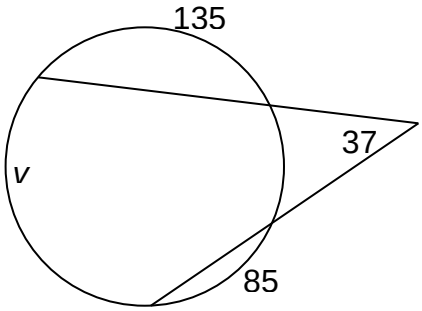


There's a rule based on similar triangles that says $12 \cdot 4 = 48 = 8w$, so $w = \frac{48}{8} = 6$.

37. How many vertices does a regular dodecahedron have?

A dodecahedron has 12 faces, each of which is a regular pentagon. 12 pentagons would have a total of $12 \cdot 5 = 60$ vertices, but every vertex of the dodecahedron is the intersection of three faces, so in reality there are only $\frac{60}{3} = 20$ vertices.

38. The figure to the right shows a circle with two intersecting secants with several angle and arc measures given in degrees. What is the value of v?



The missing arc measures $360 - 135 - 85 - v = 360 - 220 - v = 140 - v$, so we can write $\frac{v-(140-v)}{2} = 37$, then $2v - 140 = 74$, giving $2v = 214$ and finally $v = 107$.

39. What is the circumference, in meters, of a circle with an area of 2π m^2?

If the area is 2π, the radius is $\sqrt{2}$, so the diameter is $2\sqrt{2}$ and the circumference is $2\pi\sqrt{2}$.

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40. What is the measure, in degrees, of an interior angle of a regular polygon with 8 sides?

An octagon has exterior angles of $\frac{360}{8} = \frac{180}{4} = \frac{90}{2} = 45°$, so the interior angles are each $180 - 45 = 135°$.

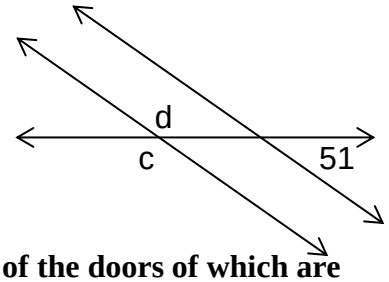
41. What is the most specific name for a triangle with two angles measuring $54°$ and $49°$?

The third angle is $180 - 54 - 49 = 180 - 103 = 77°$. The three angles are different, so the three sides will be different, so the triangle is "scalene". In addition, the three angles are all less than $90°$, so the triangle is "acute".

42. What is the most specific name that can be applied to all quadrilaterals with two pairs of opposite angles which are congruent?

Thinking about it a bit, you should realize that the answer is "parallelogram".

43. The figure to the right shows three lines, two of which are parallel, with several of the resulting angles labeled in degrees. What is the sum of *c* and *d*?



The supplement of $51°$ is $180 - 51 = 129°$. Both c and d are congruent to this angle, so their sum is $2 \cdot 129 = 258°$.

44. A cow is tied to an outside corner of a rectangular barn, all of the doors of which are closed. The barn measures 2 m by 6 m, and the cow's rope is 6 m long. What is the area, in square meters, that the cow can graze?

The cow can clearly graze $\frac{3}{4}$ of a circle with a radius of 6 m. Less clearly, the cow can wrap around a corner, thus grazing an additional $\frac{1}{4}$ of a circle with a radius of $6 - 2 = 4$ m. Thus, the answer is $\frac{3}{4} \cdot 6^2\pi + \frac{1}{4} \cdot 4^2\pi = \frac{3 \cdot 36\pi + 16\pi}{4} = \frac{(108+16)\pi}{4} = \frac{124\pi}{4} = 31\pi$.

45. What is the largest floorspace, in square meters, that can be isolated in a corner of a rectangular room by placing a screen that is four feet wide?

The screen can cut off a right triangle in the corner of the room. One might assume that symmetry is good, but you can be certain that that is the solution if you consider four screens in four rooms that share a corner; those four screens should form a square in order to enclose the maximum possible area. Each would thus enclose an area of $\frac{1}{4}4^2 = 4$.

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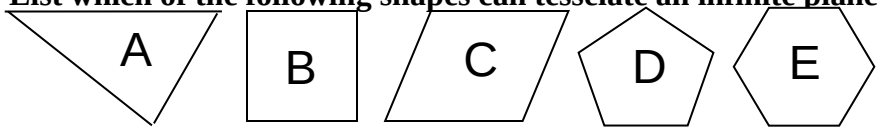
46. List which of the following shapes can tesselate an infinite plane by themselves.



Any triangle can tesselate a plane, so A works. Squares obviously work, so B is good. Parallelograms can also tesselate, so C is good. Some pentagons can tesselate a plane, but this probably-regular pentagon cannot (consider orienting them about a single vertex; it won't work), so D is bad. A regular hexagon can tesselate a plane, as can some other hexagons, so E works, for an answer of ABCE.

47. What is the largest number of regions that 5 lines can divide a plane into?

0 lines divide a plane into 1 region. 1 makes 2. 2 makes 4. 3 makes 7. Now it gets a little tricky… if you're drawing and counting, you should make a large picture so that it's easy to keep track of regions as some of them get very small. You can count as you draw; each time you bisect a region, you create one additional region. Instead of drawing, you might also notice a pattern: 1, 2, 4, 7 has differences of 1, 2, 3, … Thus, four lines should make 11 regions and five lines should make 16 regions.

48. What is the measure, in degrees, of an angle that is complementary to an 81-degree angle?

Complementary angles add up to 90°, for an answer of $90° - 81° = 9°$.

49. When the vertices of a regular polygon are labeled in clockwise order starting with A, B, C, etc., a line through D and L passes through the center of the polygon. How many sides does the polygon have?

How many letter apart are D & L? Just count on your fingers: E, F, G, H, I, J, K, L is 8, which is halfway around the polygon, so there must be $2 \cdot 8 = 16$ sides.

50. What is the measure, in degrees, of the smaller angle between the hour and minute hands of a standard 12-hour analog clock at 12:20 AM?

There are 12 hours around the dial, so they are $\frac{360°}{12} = 30°$ apart. You might think this means the hands are $\frac{12}{5} \cdot 30° = 4 \cdot 30° = 120°$ apart, but that ignores the fact that the hour hand moves a little as the minute hand moves; at 12:20 it's not pointing at the 12, it's a little bit towards the 1. How big is that little bit? It's $\frac{20}{60} \cdot 30° = \frac{1}{3} \cdot 30° = 10°$, for an answer of $120° - 10° = 110°$.

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51. A rectangular picture measuring 5 inches by 4 inches has a rectangular frame that is 3 inches wide on each side. What is the area, in square inches, of just the frame?

On each side of the picture there will be a rectangle of frame three inches wide, as well as four corners of frame that are 3-inch squares, for an area of $3(5+4+5+4)+4\cdot 3^2 = 3\cdot 18+4\cdot 9 = 54+36 = 90$.

52. Evaluate: $(\mathbf{1}-\boldsymbol{i})^{\mathbf{3}}+(\mathbf{1}+\boldsymbol{i})^{\mathbf{2}}$

Thinking of them in terms of magnitude and angle, it's clear this becomes $(-2-2i)+(2i) = -2$.

53. What are the coordinates, in the form $(\boldsymbol{x},\boldsymbol{y})$, of the center of the locus of points satisfying $\mathbf{2}\boldsymbol{x}^{\mathbf{2}}-\boldsymbol{y}^{\mathbf{2}}+\mathbf{8}\boldsymbol{x}+\mathbf{6}\boldsymbol{y}=\mathbf{100}$?

Completing the squares gives $2(x+2)^2-(y-3)^2 = 100+2\cdot 2^2-(-3)^2$. We don't care about the right side; the left is enough to know that the center is $(-2,3)$.

54. If $\boldsymbol{t}(\boldsymbol{s})=\mathbf{5}\boldsymbol{s}-\mathbf{7}$, and $\boldsymbol{r}(\boldsymbol{q})=\mathbf{2}\boldsymbol{q}+\mathbf{7}$, evaluate $\boldsymbol{r}\big(\boldsymbol{t}(-\mathbf{2})\big)$.

$r\big(t(-2)\big) = r(5(-2)-7) = r(-10-7) = r(-17) = 2(-17)+7 = -34+7 = -27$

55. Express the range of $\boldsymbol{m}(\boldsymbol{k})=\mathbf{6}-\sqrt{\mathbf{7}\boldsymbol{n}+\mathbf{1}}$ in interval notation, given that both the domain and range are subsets of the real numbers.

The outputs of this function will be 6 minus a number greater than or equal to zero, so the range will be $(-\infty, 6]$.

56. I've invested $10,000 in an account that receives 40% annual interest, compounded quarterly. How much will be in the account at the end of the first year?

Each quarter we'll get $\frac{40\%}{4} = 10\%$ interest, so after each quarter we'll have $11,000, $12,100, $13,310, and $14,641.

57. What is the sum of the roots of $\mathbf{7}\boldsymbol{v}^{\mathbf{5}}-\mathbf{3}\boldsymbol{v}^{\mathbf{4}}+\mathbf{3}\boldsymbol{v}^{\mathbf{3}}+\mathbf{7}\boldsymbol{v}^{\mathbf{2}}-\mathbf{5}\boldsymbol{v}+\mathbf{3}=\mathbf{1}$?

The sum of the roots of the polynomial is $-\frac{b}{a} = -\frac{-3}{7} = \frac{3}{7}$.

58. $(\mathbf{2}\boldsymbol{w}-\mathbf{1})^{\mathbf{6}}$ is expanded, like terms are combined, and the terms are arranged in descending degree. What is the middle term, including sign?

The first term will be w^6, and the last is w^0, so the middle term will be w^3. Specifically, this will be $\binom{6}{3}(2w)^3(-1)^3 = \frac{6!}{3!\cdot 3!}\cdot 8w^3(-1) = 20(-8w^3) = -160w^3$.

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59. Simplify: $\frac{6b^4+7b^3-10b^2+4b+8}{b+2}$

Long division requires $6b^3$, which produces $6b^4 + 12b^3$, so we need $-5b^2$, which gives $-5b^3 - 10b^2$, so we don't need a b term, and we'll end with a 4 to make $4b + 8$, for an answer of $6b^3 - 5b^2 + 4$.

60. What is the product of the three complex third roots of $8i$?

Each of these numbers will satisfy $b^3 = 8i$, which can be written as $b^3 - 8i = 0$. This is a cubic equation, the product of whose roots will be $P = (-1)^n \frac{z}{a} = (-1)^3 \frac{(-8i)}{1} = (-1)(-8i) = 8i$.

61. How many counting numbers between 33 and 50 inclusive are prime?

Even numbers won't be prime, so we'll consider the odds. 35 is a multiple of 5, 37 is prime, 39 is a multiple of 3, 41 is prime, 43 is prime, 45 is a multiple of 5, 47 is prime, and 49 is a multiple of 7, making our answer 4.

62. Express the base-9 numeral 270_9 as a base-10 numeral.

In base 9, the digits from right to left represent $9^0 = 1$s, $9^1 = 9$s, $9^2 = 81$s, etc., so $270_9 = 2 \cdot 81 + 7 \cdot 9 + 0 \cdot 1 = 162 + 63 = 225$.

63. Express the base-10 number 478_{10} as a base-8 numeral.

In base 8, the digits from right to left represent 1s, 8s, 64s, and 512s. There are 7 64s in 478, leaving $478 - 448 = 30$. There are 3 8s in 30, leaving $30 - 24 = 6$, for an answer of 736_8.

64. Express the binary number 101101_2 as a base-4 numeral.

Each pair of digits in base 2 corresponds to one digit in base 4, so the rightmost digit will be $0 \cdot 2 + 1 \cdot 1 = 0 + 1 = 1$, the next digit will be $1 \cdot 2 + 1 \cdot 1 = 2 + 1 = 3$, and the first digit will be $1 \cdot 2 + 0 \cdot 1 = 2 + 0 = 2$, for an answer of 231_4.

65. What is the prime factorization, in exponential form, of 672?

$672 = 2^3 \cdot 84 = 2^5 \cdot 3 \cdot 7$

66. How many positive integers are factors of 756?

$756 = 2^2 \cdot 189 = 2^2 \cdot 3^2 \cdot 21 = 2^2 \cdot 3^3 \cdot 7^1$, so there are $(2 + 1)(3 + 1)(1 + 1) = 3 \cdot 4 \cdot 2 = 24$ factors.

67. What is the greatest common factor of 48 and 318?

$48 = 2^4 \cdot 3$, and $318 = 2 \cdot 3 \cdot 53$, so the greatest common factor is $2 \cdot 3 = 6$.

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68. Isaac begins writing the counting numbers in order on the sidewalk, circling every multiple of 80 in purple chalk. Rio follows after him, circling 17 and every 15th number thereafter (32, 47, etc.) with red chalk. What is the smallest positive difference between a red number and a purple number?

The greatest common factor of 80 and 15 is 5, so no matter how many 15s or 80s we add, we'll always be changing the difference between circled numbers by a multiple of 5. Their initial separation was $17 - 0 = 17$, so it could be changed to 12, 7, 2, or -3, for an answer of 2.

69. What is the units digit when 3598^{68} is evaluated?

Only the last digit matters. $8^1 = 8$, $8 \cdot 8 = 64$ (final digit is 4), $8 \cdot 4 = 32$ (2), $8 \cdot 2 = 16$ (6), and $8 \cdot 6 = 48$ (8), starting a repeating pattern with period 4, so 8^{68} will end with the same digit as 8^4, for an answer of 6.

70. What is the sum of the first 27 terms of the arithmetic (adding or subtracting) sequence whose first three terms are 1, 5, and 9?

The common difference is $5 - 1 = 4$, so the 27th term is $1 + 26 \cdot 4 = 1 + 104 = 105$ and "outer pairs" sum to $1 + 105 = 106$, for an answer of $\frac{27}{2} \cdot 106 = 27 \cdot 53 = 1431$.

71. What is the 5th term of the geometric (multiplying or dividing) sequence whose first three terms are 7, 28, 112?

The common ratio is $\frac{28}{7} = 4$, for an answer of $112 \cdot 4^2 = 112 \cdot 16 = 1792$.

72. What is the missing term of the quadratic sequence 2, 3, 27, 74, 144, ___, ...?

The differences are 1, 24, 47, and 70, which is arithmetic with common difference 23, so the next difference should be $70 + 23 = 93$, for an answer of $144 + 93 = 237$.

73. A sequence is defined as $b_n = \frac{1}{n} - \frac{1}{n+1}$. What is the sum of b_5 through b_{18}?

The desired sum is $\left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{17} - \frac{1}{18}\right) + \left(\frac{1}{18} - \frac{1}{19}\right)$, which includes a LOT of cancelling (this kind is often called "telescoping"), and leaves $\frac{1}{5} - \frac{1}{19} = \frac{19-5}{5 \cdot 19} = \frac{14}{95}$.

74. What is the missing term of the sequence 1, 6, 15, 12, 29, 24, ___, ...?

The sequence isn't immediately obvious, so let's look at the differences: 5, 9, -3, 17, -5, … Hmmm… that's not obvious, either. Might it be two interspersed sequences? 1, 15, 29, … is an arithmetic sequence with common difference 14, for an answer of $29 + 14 = 43$. To confirm, 6, 12, 24 is a geometric sequence with common difference 2, so the interspersed theory seems sound, although we might like more terms to be sure.

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75. What is the sum of the 16 smallest positive odd numbers?

You can treat this as a generic arithmetic sequence or memorize that the sum of the n smallest odd numbers is n^2, for an answer of $16^2 = 256$.

76. What is the sum of the 28 smallest positive even numbers?

You can treat this as a generic arithmetic sequence or memorize that the sum of the n smallest even numbers is twice the sum of the n smallest counting numbers, for an answer of $28 \cdot 29 = 812$.

77. What is the sum of the 19 smallest positive perfect cubes?

You can memorize that the sum of the n smallest cubes is the square of the n smallest counting numbers, for an answer of $\left(\frac{19 \cdot 20}{2}\right)^2 = 190^2 = 36{,}100$.

78. A bag contains 5 red marbles and 9 orange marbles. When two marbles are drawn without replacement, what is the probability that the first marble drawn is orange and the second marble drawn is red?

The probabilities of the draws are $\frac{9}{14}$ and $\frac{5}{13}$ respectively, for an answer of $\frac{9}{14} \cdot \frac{5}{13} = \frac{45}{182}$.

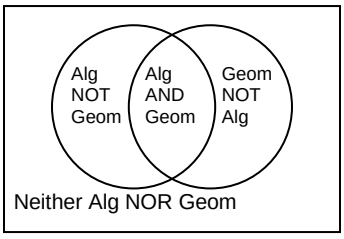
79. When three fair coins are flipped, what is the probability that they show exactly 1 tail?

There are $2^3 = 8$ ways to flip three coins, and there are three ways to get 1 tail (THH, HTH, HHT), for an answer of $\frac{3}{8}$.

80. In how many ways can 3 purple books, 1 blue book, 2 green books, and 1 yellow book be arranged next to one another on a shelf if books of the same color must all be next to one another?

The four colors could be arranged in $4! = 24$ ways, the purples in $3! = 6$ ways, and the greens in $2! = 2$ ways, for an answer of $24 \cdot 6 \cdot 2 = 288$.

81. When 84 mathletes were surveyed, 25 said they liked algebra, 34 said they liked geometry, and 19 said they liked both. How many of the students surveyed liked neither algebra nor geometry?



Using a Venn diagram like that to the right, it's easy to see that $25 - 19 = 6$ liked ONLY algebra, so that $34 + 6 = 40$ liked at least one of algebra & geometry. This means that $84 - 40 = 44$ liked neither.

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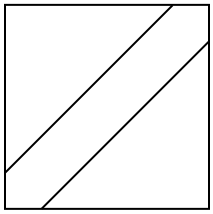
82. A circular dartboard has a radius of 7 inches and a bullseye with a diameter of 1 inches. What is the probability that a dart that hits a random location on the dartboard will hit the bullseye?

The "good" area is $\left(\frac{1}{2}\right)^2 \pi = \frac{\pi}{4}$, while the total area is $7^2\pi = 49\pi$, for an answer of $\frac{\frac{\pi}{4}}{49\pi} = \frac{1}{4 \cdot 49} = \frac{1}{196}$.

83. Clark and Lois plan to meet at the Daily Planet. Each plans to arrive at a random time between 5 PM and 6 PM, wait up to ten minutes, then leave if the other person isn't there. What is the probability that they actually meet?



You can create a figure like that to the right to represent Clark's arrival times along the bottom and Lois' arrival times along the left side. The strip in the middle represents times when they arrive within ten minutes of one another and therefore meet, and the two triangles are situations where they do not meet. The probability that they meet is thus $\frac{60^2 - 50^2}{60^2} = \frac{6^2 - 5^2}{6^2} = \frac{36-25}{36} = \frac{11}{36}$.

84. When a single die is rolled, what is the expected value of the square of the number shown?

There is a $\frac{1}{6}$ chance of getting each of $1^2 = 1$, $2^2 = 4$, 9, 16, 25, and 36, for an expected value of $\frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{1}{6} \cdot 91 = \frac{91}{6}$.

85. As I'm about to turn off my light on Halloween, 5 Trick-or-Treaters show up, and I decide to give them everything I have left: 10 identical candy bars. If I don't pay any attention to fairness (all the candy might go to one Trick-or-Treater), in how many ways might I distribute the candy?

A great method for this problem is to put $5 - 1 = 4$ rocks in the candy bag, then begin drawing and giving candy to the leftmost kid. When you get a rock, switch kids and keep going, etc. There are $\binom{10+4}{4} = \binom{14}{4} = \frac{14!}{10! \cdot 4!} = \frac{14 \cdot 13 \cdot 12 \cdot 11}{4 \cdot 3 \cdot 2} = 7 \cdot 13 \cdot 11 = 1001$.

86. Evaluate: $< -2, 4, 7 > \cdot < 9, 4, -7 >$

$-2 \cdot 9 + 4 \cdot 4 + 7(-7) = -18 + 16 - 49 = -51$

87. What is the shortest distance from the point $(3, -9, 6)$ to the plane $x + 2y - 3z = 4$?

$\frac{|3+2(-9)-3 \cdot 6-4|}{\sqrt{1^2+2^2+(-3)^2}} = \frac{|3-18-18-4|}{\sqrt{1+4+9}} = \frac{|-37|}{\sqrt{14}} = \frac{37\sqrt{14}}{14}$

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88. What is the range of the data set {35, 15, 25, 41}?

$41 - 15 = 26$

89. What is the mean of the median, mode, and range of the data set {3, 9, 95, 38, 3}?

The median is 9, the mode is 3, and the range is $95 - 3 = 92$, so the desired mean is $\frac{92+9+3}{3} = \frac{104}{3}$.

90. The 12 blue-eyed students in a class averaged 87% on a test, while the 16 brown-eyed students averaged 31%. What was the class average, as a percentage? Note: each student has either blue or brown eyes, but not both.

We need the weighted average: $\frac{12 \cdot 87 + 16 \cdot 31}{12+16} = \frac{3 \cdot 87 + 4 \cdot 31}{3+4} = \frac{261+124}{7} = \frac{385}{7} = 55$.

91. If Set A is {18, 53, 57, 23, 87, 12, 83, 24, 73, 45} and Set B is {84, 56, 12, 65, 38, 94, 80, 95, 27, 98}, what is $A \cap B$?

It turns out that 12 is the only element in both sets, and thus {12} is the intersection.

92. If Set C is {9, 3, 5, 8, 7} and Set D is {9, 1, 7, 8, 2, 3, 5}, how many subsets of Set D are supersets of Set C?

D contains all of C, plus 1 and 2, either of which can be in a superset or not, for an answer of $2^2 = 4$.

93. In the cryptarithm below, all instances of a particular letter represent the same digit (0-9), and different letters represent different digits. For example, if one A is a 7, all As are 7s and no Bs are 7s. What is the largest possible value of the three-digit number ABC?

$$\begin{array}{r} AB \\ -BC \\ \hline A \end{array}$$

For a large answer, we'd like A to be 9, so that the problem becomes 90-something minus something else equals 9. The something else must be 80-something, so now the problem is 98 minus 80-something equals 9, which requires C to be 9, but A is 9, so this is impossible. Next best would be $A = 8$, so that the problem becomes 80-something minus something else equals 8. The something else must be 70-something, so now the problem is 87 minus 70-something equals 8, which requires C to be 9, which works and gives an answer of 879.

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94. In the puzzle to the right, different one-digit numbers (1-9) can be substituted for A, B, C, and D to make all four equations (two across and two down) true. What is the largest possible product of A, B, C, and D?

A	+	B	=	13
-		/		
C	x	D	=	8
=		=		
1		4		

B/D can be 4/1 or 8/2, so that C can be 8 or 4 in those cases. A-C would thus make A 9 or 5 in those cases, while A+B also gives 9 or 5 in those cases. Thus, there are two possible solutions: $9 \times 4 \times 8 \times 1 = 288$ or $5 \times 8 \times 4 \times 2 = 320$, so the answer is 320.

95. If $\sin x = \frac{2}{3}$ and $\frac{\pi}{2} < x < 2\pi$, what is the value of $\cos(x)$?

$\cos(x) = \pm\sqrt{1 - \sin^2 x} = \pm\sqrt{1 - \left(\frac{2}{3}\right)^2} = \pm\sqrt{1 - \frac{4}{9}} = \pm\sqrt{\frac{5}{9}} = \pm\frac{\sqrt{5}}{3}$, so the only question is whether we want the positive or negative value. The value of $\sin x = \frac{2}{3}$ shows that x is in the first or second quadrant, and the restriction on x narrows that to the second quadrant, so that the answer is $-\frac{\sqrt{5}}{3}$.

96. What is the area, in square meters, of a triangle with sides measuring 5 m, 9 m, and 8 m?

Heron's Formula gives $\sqrt{11 \cdot 2 \cdot 3 \cdot 6} = 6\sqrt{11}$.

97. Express $4 - 4i\sqrt{3}$ in $re^{i\theta}$ form, where $-\pi < \theta \leq \pi$.

In the complex plane, the distance (r) of this point from the origin is $\sqrt{4^2 + \left(4\sqrt{3}\right)^2} = \sqrt{16 + 48} = \sqrt{64} = 8$. The angle (θ) has a tangent of $-\frac{4\sqrt{3}}{4} = -\sqrt{3}$, so it is coterminal to $\frac{2\pi}{3}$ or $\frac{5\pi}{3}$. Because $y < 0$, we're looking at $\frac{5\pi}{3}$, but to fall within the given range we'll need a smaller coterminal value, $-\frac{\pi}{3}$, for an answer of $8e^{-\frac{\pi}{3}i}$.

98. Evaluate: $\lim_{n\to\infty} \frac{2n^2+9n-87}{5n^2-3n+1}$

In both the numerator and the denominator, as x goes to ∞ the x^2 term becomes infinitely larger than the others, so they can be ignored. Thus, the answer is $\lim_{n\to\infty} \frac{2n^2}{5n^2} = \lim_{n\to\infty} \frac{2}{5} = \frac{2}{5}$.

99. If $f(t) = t^2 \ln t$, evaluate $f''(e)$.

$f'(\mathrm{t}) = t^2 \cdot \frac{1}{t} + 2t \ln t = t + 2t \ln t$, so $f''(t) = 1 + 2t \cdot \frac{1}{t} + 2 \ln t = 1 + 2 + 2 \ln t = 3 + 2 \ln t$, and $f''(e) = 3 + 2 \ln e = 3 + 2 \cdot 1 = 3 + 2 = 5$.

100. If a function satisfies $\frac{dy}{dx} = y + x$ and passes through the point $(8, 2)$, estimate the value of y when $x = 7.9$ using a single step of Euler's method.

$\frac{dy}{dx} = 8 + 2 = 10$ at the point $(8,2)$. $x = 7.9$ is $8 - 7.9 = .1$ to the left of 8, so it will be $.1 \cdot 10 = 1$ below 2, for an answer of $2 - 1 = 1$.