# 2014 Fall Startup Event
# Solutions

1. $1000 - 99 = 901$

2. $245 \times 97 = 245 \times 100 - 245 \times 3 = 24500 - 735 = 23765$

3. $4002 \div 46 = 4600 \div 46 - 598 \div 46 = 100 - 460 \div 46 - 138 \div 46 = 100 - 10 - 3 = 87$

4. $251 + 25.1 + 2.51 = 278.61$ by aligning decimal points.

5. $\frac{60}{100}x = 48$ becomes $x = 48 \times \frac{100}{60} = 48 \times \frac{5}{3} = 16 \times 5 = 80$.

6. $-(-1)(-2) - (-3) \div (-4) = -2 - \frac{3}{4} = -\frac{8}{4} - \frac{3}{4} = -\frac{11}{4}$

7. There are 60 seconds in a minute, and 60 minutes in an hour, for an answer of $60 \times 60 \times \frac{7}{2} = 60 \times 30 \times 7 = 1{,}800 \times 7 = 12{,}600$.

8. $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}$

9. $9 \times 8^2 \div (7 + 6 - 5) = 9 \times 8^2 \div 8 = 9 \times 8 = 72$

10. $\sqrt{525} = \sqrt{25 \times 21} = 5\sqrt{21}$

11. $1001^2 - 999^2 = (1001 - 999)(1001 + 999) = 2 \times 2000 = 4000$

12. $9 - 8z = 57$ becomes $8z = 9 - 57 = -48$, giving $z = -6$.

13. $2(3 - y) = 4(5y + 6)$ becomes $3 - y = 2(5y + 6) = 10y + 12$, then $-9 = 11y$, giving $y = -\frac{9}{11}$.

14. $x^2 - 6x + 8 = 0$ factors to $(x - 4)(x - 2) = 0$ with roots of 2 and 4.

15. The fill rate is $\frac{1}{8}$ tub per minute, and the drain rate is $\frac{1}{12}$ tub per minute, for a combined rate of $\frac{1}{8} - \frac{1}{12} = \frac{3}{24} - \frac{2}{24} = \frac{1}{24}$ tub per minute, so that it will take 24 minutes to fill the tub with the drain open.

16. The number of buckets is doubling from five to ten; if this were all that changed, the number of spigots would double from three to six. The number of minutes is divided by four, so the number of spigots must be *multiplied* by four to compensate for this, for an answer of $4 \times 6 = 24$.

17. The total amount of acid is $10 + .5 = 10.5$ liters, while the total volume is $10 + 5 = 15$, so that the fraction that is acid is $\frac{10.5}{15} = \frac{105}{150} = \frac{21}{30} = \frac{7}{10} = 70\%$.

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18. Because the numbers sum to 84, the numbers must be equidistant from $\frac{84}{2} = 42$, such as 0 & 84 or 40 & 44. Because the numbers differ by 36, they must each be $\frac{36}{2} = 18$ from 42, so that the smaller number is $42 - 18 = 24$.

19. The $x$-intercept is on the $x$-axis, where $y = 0$, so $3x - 4(0) = 3x = 24$, so $x = 8$ for an answer of $(8,0)$.

20. The slope of the line $3x + 4y = 24$ is $-\frac{A}{B} = -\frac{3}{4}$, so the slope of a perpendicular line would be the negative reciprocal, $\frac{4}{3}$.

21. $\sqrt{4 - (-2)^2 + (-7 - 3)^2} = \sqrt{6^2 + 10^2} = 2\sqrt{3^2 + 5^2} = 2\sqrt{9 + 25} = 2\sqrt{34}$

22. $\frac{|Ax+By+C|}{\sqrt{A^2+B^2}} = \frac{|1+2+2|}{\sqrt{1^2+1^2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

23. $y = 3x + 1$ goes through $(0,1)$ and heads up & right and down & left, so that it will avoid quadrant IV.

24. The axis of symmetry is $x = -\frac{b}{2a} = -\frac{-28}{2\times 2} = \frac{28}{4} = 7$.

25. The vertex is on the axis of symmetry, $x = -3$, so $y = (-3)^2 + 6(-3) - 11 = 9 - 18 - 11 = -20$, for an answer of $(-3, -20)$.

26. When two-digit numbers are reversed, the increase or decrease is always nine times the difference between the two digits. Our difference of 72 means we'd like a difference between digits of $\frac{72}{9} = 8$, so that our number must be 19.

27. His path can be thought of as four rectangles, two measuring 30 by 1 and two measuring 18 by 1, for an area of $60 + 36 = 96$.

28. The 20 kg costs $D$ dollars, which is $100D$ cents. The $K$ kg should cost $\frac{K}{20}$ times as much, which would be $\frac{K}{20} \times 100D = 5KD$.

29. 120 Wombats is $40 \times 3$ Wombats, so could be exchanged for $40 \times 4 = 160$ Vultures. The 160 Vultures is $16 \times 10$ Vultures, so could be exchanged for $16 \times 21 = 336$ Slugs.

30. $\frac{q}{1+q} = \frac{2-q}{5-q}$ becomes $5q - q^2 = 2 + q - q^2$, then $4q = 2$, giving $q = \frac{1}{2}$.

31. We can write $P = 3O$ and $P + 4 = 2(O + 4) = 2O + 8$, then substitute to get $3O + 4 = 2O + 8$, so that $O = 4$ and thus $P = 3 \times 4 = 12$.

32. FOIL gives $(n + 3)(2 - 5n) = 2n - 5n^2 + 6 - 15n = -5n^2 - 13n + 6$.

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33. $A = \frac{1}{2}hw = \frac{1}{2} \times 3 \times 4 = 2 \times 3 = 6$

34. Each of the three sides is 5, for a perimeter of $3 \times 5 = 15$.

35. We can draw an altitude to the side measuring 6, creating two right triangles with legs of 3 and hypotenuses of 7. The Pythagorean Theorem gives the altitude as $\sqrt{7^2 - 3^2} = \sqrt{49 - 9} = \sqrt{40} = 2\sqrt{10}$, so that the area is $A = \frac{1}{2}hw = \frac{1}{2} \times 6 \times 2\sqrt{10} = 6\sqrt{10}$.

36. If there's an obtuse angle, the triangle is called "obtuse".

37. A polygon with four sides is called a "quadrilateral".

38. Like a rectangle, a parallelogram's perimeter is $2a + 2b = 2(a + b) = 2(8 + 9) = 2 \times 17 = 34$.

39. $A = \pi r^2 = \pi \times 10^2 = 100\pi$

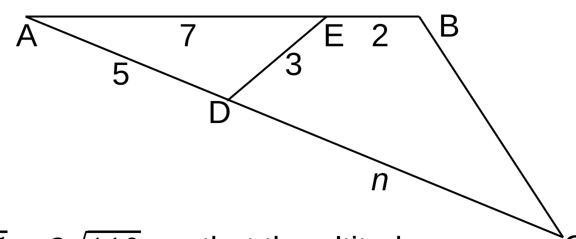
40. $V = \frac{1}{3}s^2h = \frac{1}{3} \times 9^2 \times 8 = 3 \times 9 \times 8 = 27 \times 8 = 216$

41. A seven-sided polygon is called a "heptagon" or a sometimes "septagon" (a Latin/Greek mashup).

42. The ratio of the areas is $\frac{90}{40} = \frac{9}{4}$. This is the square of the ratio of the dimensions, which would be $\sqrt{\frac{9}{4}} = \frac{3}{2}$, so that the perimeter of the larger pentagon will be $\frac{3}{2} \times 30 = 3 \times 15 = 45$.

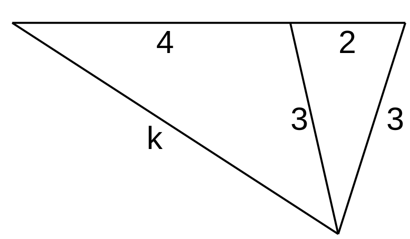
43. The volume of a cube is $s^3$, so that $s = \sqrt[3]{512} = 8$. The surface of the cube is six squares, so the area is $6 \times 8^2 = 6 \times 64 = 384$.

44. $\Delta AED$ is similar to $\Delta ACB$, so we can write $\frac{5}{9} = \frac{7}{5+n}$. Cross-multiplying gives $25 + 5n = 63$, which becomes $5n = 63 - 25 = 38$, giving $n = \frac{38}{5}$.

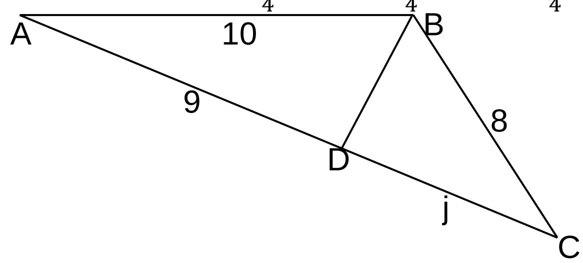



45. Heron's Formula gives $A = \sqrt{11 \times 2 \times 4 \times 5} = 2\sqrt{110}$, so that the altitude to the sides measuring 9 should satisfy $2\sqrt{110} = \frac{1}{2} \times 9a$, giving $a = \frac{4\sqrt{110}}{9}$.

46. Stewart's Theorem gives $3^2 \times 4 + k^2 \times 2 = 3^2 \times 6 + 2 \times 4 \times 6$, which becomes $36 + 2k^2 = 54 + 48 = 102$, then $2k^2 = 66$ and $k^2 = 33$, giving $k = \sqrt{33}$.

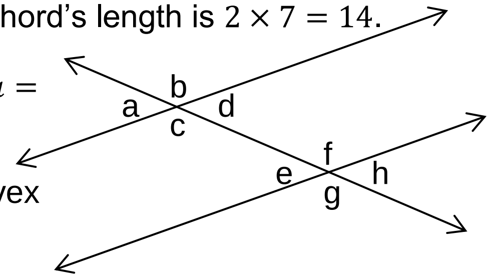


47. The goat can graze $\frac{3}{4}$ of a circle with radius 8, as well as $\frac{1}{4}$ of a circle with a radius of $8-4=4$ and $\frac{1}{4}$ of a circle with a radius of $8-6=2$, which is $\frac{3}{4}\times 64\pi+\frac{1}{4}\times 16\pi+\frac{1}{4}\times 4\pi=48\pi+4\pi+\pi=53\pi$.




48. We can write $\frac{10}{9}=\frac{8}{j}$, then cross-multiply to get $10j=72$, which gives $j=\frac{72}{10}=\frac{36}{5}$.

49. Have of the chord in question is a leg of a right triangle with a hypotenuse and other leg that are $R$ and $r$, respectively, so we can write $h^2+r^2=R^2$, then $h^2=R^2-r^2$. By a strange coincidence, the area of the annulus between the two circles is $49\pi=R^2\pi-r^2\pi=\pi(R^2-r^2)$, so that $49=R^2-r^2$. Combining these two equations gives $h^2=49$, then $h=7$, so that the chord's length is $2\times 7=14$.




50. $f+g+h=f+180=(180-a)+180=360-a=360-57=303$

51. The formula for the number of diagonals in a convex polygon is $\frac{n(n-3)}{2}=\frac{9\times 6}{2}=9\times 3=27$.

52. If the three lines bound a triangle, the regions created are the triangle, one outside each side, and one outside each vertex, for a total of $1+3+3=7$.

53. At 6:00 the angle is $180°$. In ten minutes, the minute hand moves $\frac{1}{6}\times 360=60°$ and the hour hand moves $\frac{1}{6}\times 30=5°$, so that at 6:10 the smaller angle is $180-60+5=125°$.

54. If there are 216 cubes, then the large cube was cut into $\sqrt[3]{216}=6$ sections. Any cube that is part of the four inner sections in all dimensions ($4^3=64$ cubes) will have no red paint, so that $216-64=152$ cubes have some red paint.

55. $i(2-3i)(4i+5)=i(-7i+22)=7+22i$

56. $2i^3+5i^6-9i^{10}=-2i-5+9=4-2i$

57. Starting to complete the squares, $x^2-6x$ is part of $(x-3)^2$ and $4y^2+4y$ is part of $4\left(y+\frac{1}{2}\right)^2$, so the center is at $\left(3,-\frac{1}{2}\right)$.

58. Because $2^8=256$, $\log_2 256=8$

59. $7^1=7$, $7^2=49$, and $7^3=343$, with the latter being big enough to push $g>1000$, so $h-6=3$, giving $h=9$.

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60. $f$ is $\frac{24}{6} = 4$ times $g$, so when $g = 18$, $f = 4 \times 18 = 72$.

61. The domain is the inputs ($c$) that don't "break" the function. The square root is what could be broken, if it tried to take the square root of a negative number. Thus, our domain is $30 + c - c^2 \geq 0$, which factors to $(5 + c)(6 - c) \geq 0$, which happens when $c \leq 6$ or $c \geq -5$, which is $[-5,6]$ in interval notation.

62. There are three 20 second intervals in each minute, so there are $5 \times 3 = 15$ in five minutes, leaving $\left(\frac{1}{2}\right)^{15} \times 4096 \times 1000 = \left(\frac{1}{2}\right)^{3} \times 1000 = 125$ g of the sample.

63. There will be terms from $a^{45}b^0$ to $a^0b^{45}$, which is $45 + 1 = 46$ terms.

64. $64^{-\frac{5}{2}} = 8^{-5} = \frac{1}{8^5} = \frac{1}{2^{15}} = \frac{1}{1024 \times 32} = \frac{1}{32768}$

65. $123_6 = 1 \times 36 + 2 \times 6 + 3 = 36 + 12 + 3 = 51$

66. In base four, the digits represent 1's, 4's, 16's, 64's, etc. There are three 64's in 234, leaving 42. There are two 16's in 42, leaving 10. There are two 4's in 10, leaving 2. Thus, $234_{10} = 3222_4$.

67. You can do arithmetic in any base without converting, although division is hard because it helps to have your multiplication facts memorized, which I only have in base 10. Adding $456_7$ and $542_7$, we start with the units digit. $6 + 2 = 8 = 11_7$, so we write a 1 and carry a 1. In the sevens digit, $5 + 4 + 1 = 10 = 13_7$, so we write a 3 and carry a 1. In the 49's digit, $4 + 5 + 1 = 10 = 13_7$, so we write a 3 and carry a 1, getting $1331_7$.

68. $684 = 2^2 \times 171 = 2^2 \times 3^2 \times 19$

69. $440 = 2^2 \times 2 \times 5 \times 11 = 2^3 \times 5^1 \times 11^1$, so the sum of the factors is $(1 + 2 + 4 + 8)(1 + 5)(1 + 11) = 15 \times 6 \times 12 = 90 \times 12 = 1080$.

70. $1200 = 2^2 \times 5^2 \times 2^2 \times 3 = 2^4 \times 3^1 \times 5^2$. Any multiple of 8 must have $2^3$ in its prime factorization, so the factors of 1200 can have 0 or 1 extra 2 (2 choices), 0 or 1 3 (two choices), and 0 or 1 or 2 5's (three choices). Thus, there are $2 \times 2 \times 3 = 12$ such numbers.

71. 24 and 45 have a GCF of 3, so the LCM will be $\frac{24 \times 45}{3} = 8 \times 45 = 360$.

72. There are $99{,}999 - 9{,}999 = 90{,}000$ five-digit numbers, and $4 \times 5 \times 5 \times 5 \times 5 = 2{,}500$ that have only even digits, leaving $90{,}000 - 2{,}500 = 87{,}500$ that contain at least one odd digit.

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73. $2^1$ ends in 2, $2^2$ ends in 4, $2^3$ ends in 8, $2^4$ ends in 6, $2^5$ ends in 2, $2^6$ ends in 4,etc. This pattern repeats every four terms, so $2^8$ ends in 6, $2^{12}$ ends in 6, and $2^{16}$ ends in 6, so $2^{15}$ ends in 8.

74. The sixth term is five terms away from the first, so will be $11 \times 2^5 = 11 \times 32 = 352$.

75. The differences are 3, 5, 7, 9, ?, ?, 15, 17, …, the missing terms of which appear to be 11 & 13, making the missing term of the original sequence $31 + 11 = 42 = 55 - 13$.

76. A harmonic sequence is the reciprocals of an arithmetic sequence, so our sequence goes $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \dots$

77. The sum from 1 to 79 is $\frac{79(79+1)}{2} = 79 \times 40 = 80 \times 40 - 40 = 3160$.

78. The twelve smallest odd counting numbers add up to $12^2 = 144$.

79. The eight smallest positive perfect cubes add up to the square of the sum of the eight smallest counting numbers, which is $\left(\frac{8\times 9}{2}\right)^2 = (4 \times 9)^2 = 36^2 = 1296$.

80. There are 26 red cards, and 6 other face cards (J, Q, K in clubs & spades), for a probability of $\frac{32}{52} = \frac{8}{13}$.

81. There are $2^5 = 32$ ways to flip five coins, and $5c3 = \frac{5\times 4}{2} = 5 \times 2 = 10$ ways to get three heads, for a probability of $\frac{10}{32} = \frac{5}{16}$.

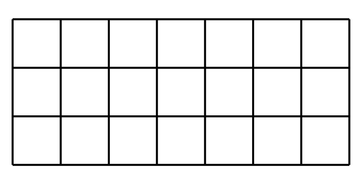
82. Nine can be rolled as 6&3 two ways and 5&4 two ways, for a total of four ways. There are $6 \times 6 = 36$ ways to roll two dice, for a probability of $\frac{4}{36} = \frac{1}{9}$.

83. $12\ c\ 5 = \frac{12\times 11\times 10\times 9\times 8}{5\times 4\times 3\times 2} = 11 \times 9 \times 8 = 11 \times 72 = 792$

84. The two artists can be arranged in $2! = 2$ ways, Huey can be arranged in $4! = 24$ ways, and Ms. Dion can be arranged in $5! = 120$, for a total of $2 \times 24 \times 120 = 24 \times 240 = 5760$ ways.

85. There are $36 - 9 = 27$ people who drank something, 25 of which had Poke, so that means $27 - 25 = 2$ of them had ONLY Cepsi. The other $16 - 2 = 14$ Cepsi drinkers also had Poke, meaning they drank both.

86. Of the ten segments on a path, three must go down, so there are $10c3 = \frac{10\times 9\times 8}{3\times 2} = 10 \times 3 \times 4 = 120$ ways.



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87. Put the breadcrumbs and $4 - 1 = 3$ pebbles in a bag and start drawing. Give breadcrumbs to the first duck until you get to the first pebble (toss it aside!), then give breadcrumbs to the second duck until you get to the second pebble, etc. There are $13c3 = \frac{13 \times 12 \times 11}{3 \times 2} = 13 \times 2 \times 11 = 2 \times 143 = 286$ ways to do this.

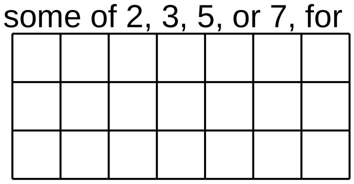
88. $< 9, -3 > - < 1,2 > = < 8, -5 >$

89. $\begin{vmatrix} 4 & 3 \\ 2 & w \end{vmatrix} = 4w - 6 = 0$ becomes $4w = 6$, giving $w = \frac{6}{4} = \frac{3}{2}$.

90. Eliminating the 2 & 14, 3 & 9, 4 & 9, 6 & 9, and 6 & 7 leaves a single 7.

91. $U^c$ is numbers without an odd digit (all evens), so $V \cap U^c$ is two-digit multiples of six with only even digits, so is 24, 42, 48, 60, 66, and 84, which is 6 elements.

92. The primes less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. Of these, we need exactly one of 13, 17, or 19, and any of 2, 3, 5, or 7. There are $3c1 = 3$ ways to pick one of 13, 17, or 19, and there are $2^4 = 16$ ways to choose some of 2, 3, 5, or 7, for an answer of $3 \times 16 = 48$.



93. There are $3 \times 7 = 21$ 1x1 squares, $2 \times 6 = 12$ 2x2 squares, and $1 \times 5 = 5$ 3x3 squares, for a total of $21 + 12 + 5 = 38$.

94. This kind of problem is typically trial and error. $2 \times 3 - 4 = 2$ and $2 \times (4 - 3) = 2$, with other solutions possible.

95. The cotangent is the reciprocal of the tangent, so it is the adjacent side over the opposite side. The smallest angle is opposite the smallest side, so the answer is $\frac{4}{3}$.

96. In the third quadrant, the cosine will be negative, and the magnitude will be $\sqrt{1 - \sin^2} = \sqrt{1 - \frac{1}{16}} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$, for an answer of $-\frac{\sqrt{15}}{4}$.

97. $\sin(2r) = 2 \sin r \cos r = 2 \sin r \times \frac{1}{3}$, and $\sin r = \pm \sqrt{1 - \cos^2 r} = \pm \sqrt{1 - \frac{1}{9}} = \pm \sqrt{\frac{8}{9}} = \pm \frac{2\sqrt{2}}{3}$, for an answer of $2 \times \left(-\frac{2\sqrt{2}}{3}\right) \times \frac{1}{3} = -\frac{4\sqrt{2}}{9}$.

98. The denominator has an $r^7$ term, while the numerator only has an $r^5$ term, so the denominator will get large faster than the numerator, making the limit 0.

99. $q(p) = 5(7p + 9)^{\frac{1}{2}}$, so $q'(p) = \frac{1}{2} \times 5(7p + 9)^{-\frac{1}{2}} \times 7 = \frac{35}{2}(7p + 9)^{-\frac{1}{2}}$, so $q'(0) = \frac{35}{2} \times 9^{-\frac{1}{2}} = \frac{35}{2} \times \frac{1}{3} = \frac{35}{6}$.

100. $\int_2^4 \left(6x + \frac{8}{x}\right) dx = 3x^2 + 8\ln x|_2^4 = 3(4^2 - 2^2) + 8(\ln 4 - \ln 2) = 3 \times 12 + 8\ln\frac{4}{2} =$
$36 + 8\ln 2$