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Solutions

1. Use the standard subtraction algorithm.

2. Use the standard multiplication algorithm.

3. Use the standard division algorithm.

4. The whole parts may be added to yield $5 + 2 = 7$. The factional parts may be added once they have a common denominator: $\frac{3}{4} + \frac{1}{6} = \frac{9}{12} + \frac{2}{12} = \frac{11}{12}$.

5. $\frac{65}{100} \times 340 = \frac{65}{10} \times 34 = \frac{65}{5} \times 17 = 13 \times 17 = 221$

6. $-1 - 4(-5) - (-6)(-7) = -1 + 20 - 42 = -43 + 20 = -23$

7. $3^6 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 9 \times 9 \times 9 = 81 \times 9 = 729$

8. $\frac{9!}{5! \times 4! \times 3!} = \frac{9\times8\times7\times6\times5\times4\times3\times2}{5\times4\times3\times2\times4\times3\times2\times3\times2} = \frac{9\times8\times7\times6}{4\times3\times2\times3\times2} = \frac{9\times8\times7}{4\times3\times2} = \frac{9\times7}{3} = 3 \times 7 = 21$

9. $9 + 8 \times 7 - 5 \times 4^3 = 9 + 56 - 5 \times 64 = 65 - 320 = -255$

10. $19^2 - 16^2 = (19 + 16)(19 - 16) = 35 \times 3 = 105$

11. $7 + 8 + 31 = 46$

12. The final result is 44, so the prior result must have been $44 \times 4 = 176$, so that the secret number is $176 + 14 = 190$.

13. $6b - 5 = 37$ becomes $6b = 42$, giving $b = 7$.

14. $3c + 4 = 5c + 6$ becomes $-2 = 2c$, giving $-1 = c$.

15. The quadratic formula gives $\frac{5 \pm \sqrt{25+64}}{2\times2} = \frac{5 \pm \sqrt{89}}{4}$.

16. Adding the two equations gives $4f = 16$, so that $f = 4$. Substituting produces $3 \times 4 + 5g = 7$, then $12 + 5g = 7$ and $5g = -5$, so that $g = -1$.

17. $distance = rate \times time$, so $30 = 100t$ (in hours), giving $t = \frac{30}{100} = \frac{3}{10}$, which is really $\frac{3}{10} \times 60 = 3 \times 6 = 18$ minutes.

18. In hours, $30 = \frac{2}{3}r$ becomes $r = 30 \times \frac{3}{2} = 15 \times 3 = 45$.

19. Each number is $\frac{48}{2} = 24$ away from $\frac{84}{2} = 42$ so that the smaller number is $42 - 24 = 18$.

20. The slope of a parallel line is the same as the slope of this line, which is $-\frac{A}{B} = -\frac{3}{-5} = \frac{3}{5}$.

21. $\sqrt{\left(4-(-2)\right)^2 + (-7-2)^2} = \sqrt{6^2+9^2} = 3\sqrt{2^2+3^2} = 3\sqrt{4+9} = 3\sqrt{13}$

22. Substituting produces $3x - 4(2x-7) = 13$, which becomes $3x - 8x + 28 = 13$, then $-5x = -15$, giving $x = 3$. Substituting *this* gives $y = 2 \times 3 - 7 = 6 - 7 = -1$.

23. This parabola goes sideways, so the axis of symmetry is $y = -\frac{b}{2a} = -\frac{-36}{2\times 3} = \frac{36}{6} = 6$.

24. The axis of symmetry is $x = -\frac{6}{2} = -3$. Substituting gives $y = (-3)^2 + 6(-3) + 5 = 9 - 18 + 5 = -4$, for an answer of $(-3, -4)$.

25. $0 = 2x^2 + 5x - 12$ factors to $0 = (2x-3)(x+4)$, with roots of $\frac{3}{2}$ and -4, for an answer of $(-4, 0)$.

26. If all 17 heads belonged to humans, there would be $17 \times 2 = 34$ heads, which is $52 - 34 = 18$ too few. For each human we turn into a dog, we gain $4 - 2 = 2$ legs, so we need to turn $\frac{18}{2} = 9$ humans into dogs.

27. She'll still be eight years older when she's three times his age, so they'll be 4 and 12 (in two years). Similarly, they'll be 8 and 16 four years later, which is six years from now.

28. $(3q+2)(q-4) = 3q^2 - 12q + 2q - 8 = 3q^2 - 10q - 8$

29. $r(-3) = 5(-3)\left((-3)+1\right) = -15(-2) = 30$

30. $3 \times 17 = 51$

31. The triangle cannot be 18-18-41, so it must be 41-41-18, and can thus be divided into two right triangles each with a hypotenuse of 41 and a leg of 9. The Pythagorean Theorem gives a height of $\sqrt{41^2 - 9^2} = \sqrt{1681 - 81} = \sqrt{1600} = 40$, so that the area is $\frac{1}{2} \times 18 \times 40 = 9 \times 40 = 360$.

32. It's not possible to have more than one angle that is more than $90°$, so it is not possible to have fewer than two angles that are less than $90°$, so there was a bit more information here than you really needed. A triangle with an obtuse angle is called "obtuse".

33. $A = \pi r^2 = \pi \times 8^2 = 64\pi$

34. A nonagon has nine sides, so its perimeter will be $9 \times 13 = 117$.

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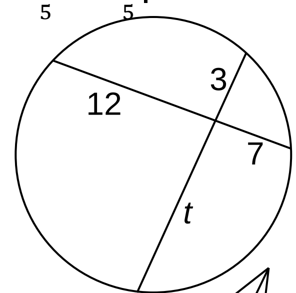
35. If the area is 289π, then then radius is $\sqrt{289} = 17$, so that the circumference is $C = 2\pi r = 2\pi \times 17 = 34\pi$.

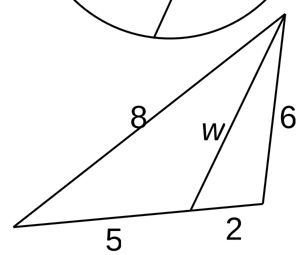
36. First of all, the largest possible area of a quadrilateral with a perimeter of 16 is 16 (a square), so the perimeter of 16 corresponds to the area of 14. Because the areas are in the ratio $\frac{56}{14} = 4$, the perimeters will be in the ratio $\sqrt{4} = 2$, making the other perimeter $2 \times 16 = 32$.

37. Because $V = \frac{1}{3}Bh$, we can write $96 = \frac{1}{3} \times 20h$, so that $h = \frac{96\times3}{20} = \frac{24\times3}{5} = \frac{72}{5}$.



38. When two chords intersect, the product of the parts of one must be equal to the product of the parts of the other, so that $12 \times 7 = 3t$, giving $t = 4 \times 7 = 28$.

39. We can write $\frac{x-14}{2} = 41$, which becomes $x - 14 = 82$, so that $x = 96$.



40. Stewart's Theorem gives $6^2 \times 5 + 8^2 \times 2 = w^2 \times 7 + 2 \times 5 \times 7$, which becomes $180 + 128 = 7w^2 + 70$, then $7w^2 = 238$ and $w^2 = 34$, for an answer of $\sqrt{34}$.

41. Vertical angles are those that are across a vertex from one another. When two lines intersect, the total of all four angles is $360°$, so the sum of the other pair is $360 - 48 = 312°$.

42. This is a common problem with two easy routes to a solution. It is easy to construct a right triangle with a hypotenuse of R and legs of r and $\frac{56}{2} = 28$, with the Pythagorean Theorem giving $R^2 - r^2 = 28^2 = 784$, so that the area between the two circles is 784π. The other solution presumes that the problem is well defined with a single answer. If so, just assume the central circle has a radius of 0, so that the chord is actually a diameter of the larger circle, and the area between the two circles is simply the larger circle, with an area of $28^2\pi$.

43. Pick a vertex and draw as many diagonals as you can. You'll find that you can't draw one to the vertex itself, nor can you draw one to either of the neighboring vertices. Thus, this vertex can have 4 diagonals drawn. Move over one vertex and find that you can draw another 4 diagonals. Keep moving over, but now you can only draw 3, then 2, and then 1, for a total of 14. You can also memorize the formula $\frac{n(n-3)}{2}$, which gives $\frac{7\times4}{2} = 7 \times 2 = 14$.

44. Of all the triangles with a 10 cm hypotenuse, the skinny ones have areas near 0, so logically the maximum area will be for an isosceles right triangle. It would have legs measuring $\frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$ and thus an area of $\frac{1}{2}\left(5\sqrt{2}\right)^2 = \frac{1}{2} \times 25 \times 2 = 25$.

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45. 0 lines divide a plane into 1 region. 1 line divides a plane into 2 regions (+1). 2 lines divide a plane into 4 regions (+2). Next comes 7 (+3), 11 (+4), 16 (+5), and 22 (+6).

46. G's supplement is $180 - G$, while the compliment is $90 - G$, for a difference of $180 - 90 = 90°$.

47. R is the 18th letter, so we need the letter that is 9 after D (the 4th letter), so the 13th letter, which is M.

48. Beginning to complete the squares gives $3(x^2 + 4x) - 2(y^2 - 10y) = 38$, so that there will be an $(x + 2)^2$ term and a $(y - 5)^2$ term, for a center at $(-2{,}5)$.

49. A parabola is the points that are equidistant from a point and a line, a hyperbola is the points that are closer to the line than the point (by a fixed ratio), and an ellipse is the points that are closer to the point than the line (by a fixed ratio).

50. $\log_8 4 = \log_8 2^2 = 2\log_8 2 = 2 \times \frac{1}{3} = \frac{2}{3}$

51. $2^{10} = 1024$, and would definitely satisfy the conditions. $2^9 = 512$, and triple that would still exceed 1098. $2^8 = 256$, but triple that won't even be 800, so the answer is 9.

52. We call this a "hidden quadratic": $9^j = (3^2)^j = 3^{2j} = \left(3^j\right)^2$, so we can factor $9^j - 30 \cdot 3^j + 81 = 0$ to $\left(3^j - 3\right)\left(3^j - 27\right) = 0$, so that 3^j could be 3 or 27, so that *j* could be 1 or 3.

53. The domain of a function is all the input values (in this case *m*) that don't break the function in any way. For $k(m) = \frac{\sqrt{49-m^2}}{2m-14}$, we can't take a square-root of a negative, and we can't divide by 0. This means that $m^2 \leq 49$ and $2m \neq 14$, so $-7 \leq m \leq 7$, and $m \neq 7$, for an answer of $[-7{,}7)$.

54. *n* was multiplied by $\frac{216}{72} = 3$, so *p* should be multiplied by 3, too, giving $p = 3 \times 27 = 81$.

55. An hour is six ten-minute half-lives, so we'll have $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$ of the sample left. 1% would be $\frac{1}{100}$, and 2% would be $\frac{2}{100} = \frac{1}{50}$, so that latter is presumably our answer. To be sure, 1.5% would be $\frac{1.5}{100} = \frac{3}{200}$, while $\frac{1}{64} = \frac{3}{192}$, making $\frac{1}{64}$ slightly larger than 1.5%, confirming that 2% is the answer.

56. The q^2 term is formed by choosing the $2q$ term twice and the 3 term twice. There are $4c2 = \frac{4!}{2! \times 2!} = \frac{4 \times 3}{2} = 2 \times 3 = 6$ ways to choose, so the q^2 term will be $6(2q)^2(3)^2 = 6 \times 4q^2 \times 9 = 216q^2$.

57. The five fifth-roots are the five roots of $x^5 = 32$, which can also be written $x^5 - 32 = 0$. The product of these roots is $(-1)^n \frac{z}{a} = (-1)^5 \times \frac{-32}{1} = 32$.

58. $\log_2 4r = \log_2 4 + \log_2 r = 2 + \frac{1}{s}$

59. 83, 89, and 97 make the answer 3.

60. $243_5 = 3 + 4 \times 5 + 2 \times 25 = 3 + 20 + 50 = 73$

61. $6^3 = 216$, which goes into 243 once, leaving 27. $6^2 = 36$ doesn't go into this at all, still leaving 27. $6^1 = 6$ goes into 27 four times, leaving 3, making the answer 1043_{10}.

62. $108 = 2 \times 54 = 2^2 \times 27 = 2^2 \times 3^3$

63. $140 = 2^2 \times 5^1 \times 7^1$, which means there are $(2+1)(1+1)(1+1) = 3 \times 2 \times 2 = 12$ factors.

64. $300 = 2^2 \times 3 \times 5^2$, so the sum of the factors will be $(1 + 2 + 4)(1 + 3)(1 + 5 + 25) = 7 \times 4 \times 31 = 28 \times 31 = 28 + 840 = 868$.

65. The hundreds & ones could be 0-0, 2-1, 4-2, 6-3, or 8-4, for five options. There are nine options for the thousands (no zeros), and ten for the tens, for a total of $5 \times 9 \times 10 = 450$.

66. $1001 \times 99 = 99099$

67. $7 \times 2^5 = 7 \times 32 = 210 + 14 = 224$

68. The differences are 3, 8, 15, ?, ?, 48, and 63. If this sequence isn't obvious (one less than perfect squares), the next differences are 5, 7, ?, ?, ?, 15. This is odd numbers, with the missing numbers being 9, 11, and 13, so that the prior missing numbers are $15 + 9 = 24$ and $24 + 11 = 35$ (double-check that $35 + 13 = 48$; it is). Thus, the missing term of the original sequence is $35 + 24 = 59$ (double-check that $59 + 35 = 94$; it is).

69. The terms of a harmonic sequence are the reciprocals of the terms of an arithmetic sequence. For this harmonic sequence, the arithmetic sequence would be $\frac{1}{7}, \frac{1}{5}, \ldots$ The common difference is $\frac{1}{5} - \frac{1}{7} = \frac{7-5}{35} = \frac{2}{35}$, so the next term is $\frac{1}{5} + \frac{2}{35} = \frac{7+2}{35} = \frac{9}{35}$, making the next term of the harmonic sequence $\frac{35}{9}$.

70. The common difference is $\frac{44-17}{9} = \frac{27}{9} = 3$, so the 71st term is $17 + 70 \times 3 = 17 + 210 = 227$.

71. If the second term is 20, the first term must be $3 \times 20 = 60$, for a sum of $\frac{60}{1-\frac{1}{3}} = \frac{60}{\frac{2}{3}} = 60 \times \frac{3}{2} = 30 \times 3 = 90$.

72. The differences are 4, 3, 7, ?, ?, 27, 44, and 71, most of which look like the original sequence! This is Fibonacci-like, with missing differences of 10 and 17, so that the missing term is 27.

73. You could add them up, or you could use the formula $\frac{n(n+1)(2n+1)}{6} = \frac{(12\times13\times25)}{6} = 2 \times 13 \times 25 = 13 \times 50 = 650$.

74. There are nine such cubes, so we can use the formula $\left(\frac{n(n+1)}{2}\right)^2 = \left(\frac{(9\times10)}{2}\right)^2 = 45^2 = 2025$.

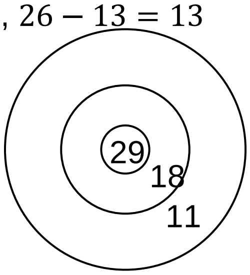
75. There are $4c1 = 4$ ways to choose the one tail, for a probability of $\frac{4}{2^4} = \frac{4}{16} = \frac{1}{4}$.

76. The sum could be 4, 8, or 12, with probabilities of $\frac{3}{36}$, $\frac{5}{36}$, and $\frac{1}{36}$, for a total of $\frac{9}{36} = \frac{1}{4}$.

77. The reds can line up in $3! = 6$ ways, the yellows have $4! = 24$ ways, and there are $2! = 2$ ways for the colors to arrange themselves (red first or yellow first), for a total of $6 \times 24 \times 2 = 12 \times 24 = 288$ ways.

78. There are $30 - 3 = 27$ members who grew one of the two. If 14 grew tomatoes, $27 - 14 = 13$ did not, so grew only peas. Of the 26 who grew peas, $26 - 13 = 13$ grew both peas and tomatoes.

79. Essentially, the question is “what is the probability that I won’t hit the bull’s-eye?”, which is $\frac{9^2-2^2}{9^2} = \frac{81-4}{81} = \frac{77}{81}$.



80. There is a $\frac{3}{8}$ probability of getting \$2 and a $\frac{5}{8}$ probability of getting \$6, for an expected value of $\frac{3}{8} \times 2 + \frac{5}{8} \times 6 = \frac{3}{4} + \frac{5\times3}{4} = \frac{18}{4} = \frac{9}{2} = 4.50$.

81. The three socks could end up with 5-0-0, 4-1-0, 3-2-0, 3-1-1, or 2-2-1 buttons, making our answer 5.

82. 4 is in a position that has a -1 involved with it, so the cofactor will be $-\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -(2 \times 9 - 3 \times 8) = -(18 - 24) = -(-6) = 6$.

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83. The "shoelace method" gives subtotals of $1 \times 3 + (-1)(-3) + (-2)0 = 3 + 3 + 0 = 6$ and $0(-1) + 3(-2) + (-3)1 = 0 - 6 - 3 = -9$. These two subtotals must be subtracted and divided by two, giving $\frac{(6-(-9))}{2} = \frac{15}{2}$. You could also construct a rectangle around this triangle and remove right triangles.

84. The median is left when the smallest and largest elements are eliminated. Getting rid of 1&9, 1&8, 3&7, and 5&6 leaves 6 as the median.

85. The mean is the average, which is $\frac{2+8+4+5+6+1}{6} = \frac{26}{6} = \frac{13}{3}$.

86. If there are at least two 67's and something 81 or larger, the smallest possible median is 67.

87. T has $\frac{900}{10} = 90$ elements, but multiples of 40 won't be in $T \cap U'$, so we lose everything from 120 to 960, which is $\frac{(960-120)}{40} + 1 = \frac{840}{40} + 1 = 21 + 1 = 22$ elements, leaving $90 - 22 = 68$ elements in $T \cap U'$.

88. There are 9 multiples of three less than 30, four of which are even. We can choose any three of the four ($4c3 = 4$ ways), and each odd number can be in or out ($2^5 = 32$ ways), for a total of $4 \times 32 = 128$ ways.

89. $9 \times 8 \div 6$ and variations work.

90. A 3-4-5 triangle is a right triangle, so the tangent is opposite over adjacent. The smallest angle is opposite the smallest side, so the tangent is $\frac{3}{4}$.

91. $\cos(150°) = -\frac{\sqrt{3}}{2}$

92. $\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{2\sqrt{2}}{3} \times \frac{4}{5} - \frac{1}{3} \times \frac{3}{5} = \frac{8\sqrt{2}-3}{15}$

93. Heron's Formula gives $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{16 \times 9 \times 4 \times 3} = 4 \times 3 \times 2\sqrt{3} = 24\sqrt{3}$.

94. The normal period of the sine function is 2π, but the coefficient of the variable inside affects that, making the period of this function $\frac{2\pi}{5}$.

95. $\text{Sin}^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

96. When we substitute $\theta = 2\pi$, we get $\frac{0}{0}$, so we'll turn to L'Hopital's Rule, taking the derivative of the numerator and denominator to get $\frac{3\cos(3\theta)}{1} = 3\cos(6\pi) = 3$.

97. This is the definition of the derivative, but L'Hopital's is probably the easiest way once we see that it's $\frac{0}{0}$. $\frac{6f^2}{1} = 6 \times 3^2 = 6 \times 9 = 54$.

98. If we let $u = x^2 + 1$, then $du = 2xdx$, so we can rewrite the integral as $\int_2^5 6u^2 du = 2u^3|_2^5 = 2(125 - 8) = 2 \times 117 = 234$.

99. The region in question is a triangle, so the volume of rotation is a cone with a volume of $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 3^2 \times 6 = 18\pi$.

100. Concavity relates to the second derivative, which is $y'' = 4 - x^2$, which will be positive for $-2 < x < 2$.