2016 Team Scramble
Solutions

Easier Problems

1. Evaluate: $\mathbf{8065 + 63857}$

The standard algorithm gives 71,922.

2. Evaluate: $\mathbf{1394940 \div 804}$

The standard algorithm gives 1735.

3. How many minutes are in three days?

$3 \times 24 \times 60 = 72 \times 60 = 4320$

4. Evaluate: $\mathbf{\frac{496}{7} \div \frac{24}{56}}$

$\frac{496}{7} \div \frac{24}{56} = \frac{496}{7} \times \frac{56}{24} = \frac{124}{1} \times \frac{8}{6} = \frac{124}{1} \times \frac{4}{3} = \frac{496}{3}$

5. Express 34567.04 in scientific notation rounded to four significant figures.

Scientific notation would be 3.456704×10^4; rounding will give 3.457×10^4.

6. Evaluate: $\mathbf{\frac{14! \cdot 9! \cdot 5!}{7! \cdot 11! \cdot 8!}}$

$\frac{14! \cdot 9! \cdot 5!}{7! \cdot 11! \cdot 8!} = \frac{14 \cdot 13 \cdot 12 \cdot 9}{7 \cdot 6} = 2 \cdot 13 \cdot 2 \cdot 9 = 52 \cdot 9 = 468$

7. Express in simplest radical form: $\mathbf{\sqrt{6804}}$

$\sqrt{6804} = \sqrt{4 \cdot 1701} = 2\sqrt{9 \cdot 189} = 2 \cdot 3\sqrt{9 \cdot 21} = 6 \cdot 3\sqrt{21} = 18\sqrt{21}$

8. Simplify by rationalizing the denominator: $\mathbf{\frac{156}{\sqrt{24} - \sqrt{11}}}$

$\frac{156}{\sqrt{24} - \sqrt{11}} \times \frac{\sqrt{24} + \sqrt{11}}{\sqrt{24} + \sqrt{11}} = \frac{156(\sqrt{24} + \sqrt{11})}{24 - 11} = 12\sqrt{24} + 12\sqrt{11} = 24\sqrt{6} + 12\sqrt{11}$

9. Write the variables in order of ascending value (e.g. BADC):
$\boldsymbol{A = 946 + 3463}$ $\boldsymbol{B = 5793 - 682}$ $\boldsymbol{C = 19 \times 26}$ $\boldsymbol{D = 67895 \div 13}$

$A \cong 900 + 3500 = 4400$, $B \cong 5800 - 700 = 5100$, $C \cong 20 \times 25 = 500$, $D \cong (65000 + 2600) \div 13 = 5000 + 200 = 5200$, so the answer is CABD.

10. What value(s) of *f* satisfy $\boldsymbol{5f - 4 = 46 + 7f}$?

This becomes $-50 = 2f$, so $f = -25$.

11. What value(s) of *g* satisfy $\boldsymbol{6g^2 - g = 15}$?

$6g^2 - g - 15 = 0$ factors to $(2g+3)(3g-5)$, with roots of $-\frac{3}{2}$ and $\frac{5}{3}$.

12. Nancy has 12 liters of a 45% acid solution that she wishes to strengthen to an 80% acid solution. If she can only do this by mixing it with a 90% acid solution, how many liters of the 90% acid solution should she add?

We're mixing 45% and 90% solutions to get an 80% solution. The 45% solution is 35 from 80%, while the 90% is 10 from 80%, so we need to mix the 45% and 90% solutions in the ratio $10:35 = 2:7$. This means we should use $12 \times \frac{7}{2} = 6 \times 7 = 42$ liters of the 90% solution.

13. Pikachu and Charmander see one another at the same moment, from a distance of 1.5 km. If Pikachu runs away at 46 mps (meters per second) and Charmander chases him at 71 mps, how many seconds will it take Charmander to catch Pikachu?

Charmander catches up at $71 - 46 = 25$ mps, so it will take him $\frac{1500}{25} = \frac{300}{5} = 60$ seconds.

14. In which quadrant does the point $(-735951, 379347)$ lie?

It's to the left and up (a long way each way!), so it's in Quadrant II.

15. What are the coordinates, in the form (x, y), of the intersection of the lines $y = 3x - 7$ and $2x - 3y = -7$?

Substituting into the second equation gives $2x - 3(3x - 7) = -7$, which becomes $2x - 9x + 21 = -7$, then $-7x = -28$, so that $x = 4$. This gives $y = 3 \cdot 4 - 7 = 12 - 7 = 5$, for an answer of $(4,5)$.

16. What is the equation for the axis of symmetry of the parabola $y = 3x^2 + 9x - 25$?

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{9}{2 \cdot 3} = -\frac{3}{2}$.

17. What are the coordinates, in the form (x, y), of the rightmost x-intercept of the parabola $y = 4x^2 + 3x - 10$?

$y = 4x^2 + 3x - 10 = (4x - 5)(x + 2)$, with zeros at $x = \frac{5}{4}$ and $x = -2$, for an answer of $\left(\frac{5}{4}, 0\right)$.

18. Professor Plum wrote an equation of the form $0 = q^2 + Aq + B$ on the board for her students to solve. Violet used the wrong value of A and got roots of -1 and 4. Lavender used the wrong value of B and got roots of 1 and -10. What were the actual roots of the equation?

Violet's answers show that $B = -1 \cdot 4 = -4$, while Lavender's show that $A = -\big(1 + (-10)\big) = 9$, so that the original equation was $0 = q^2 + 9q - 4$, with roots of $q = \frac{-9 \pm \sqrt{9^2 - 4 \cdot 1(-4)}}{2 \cdot 1} = \frac{-9 \pm \sqrt{81+16}}{2} = \frac{-9 \pm \sqrt{97}}{2}$.

19. If you can buy P pianos for $10,000, how many dimes would it take to buy 100 pianos?

Instead of buying P pianos, we're buying 100, so we should take the $10,000 price and multiply it by $\frac{100}{P}$ to get $\frac{1{,}000{,}000}{P}$ dollars. There are ten dimes per dollar, making our answer $\frac{10{,}000{,}000}{P}$.

20. A field contains llamas (four legs) and emus (two legs). If there are a total of 62 legs and 22 heads, how many llamas are there?

If all 22 animals were emus, there would be $22 \times 2 = 44$ legs, which is $62 - 44 = 18$ legs less than there really are. For each animal we convert from emu to llama, we gain 2 legs. We need to do this $18 \div 2 = 9$ times, so there are 9 llamas.

21. In the system of equations $4r + 6s - t = 67$ and $4t - 3s - 2r = 45$, what is the value of t?

If you double the second equations and add it to the first you get $7t = 157$, so that $t = \frac{157}{7}$.

22. What is the solution, in the form (w, x), of the system of equations $4w - 6x = -22$ and $7w + x = -4$?

Adding six of the second equation to the first gives $46w = -46$, so that $w = -1$. This gives $-4 - 6x = -22$, which becomes $-6x = -18$, giving $x = 3$ for an answer of $(-1,3)$.

23. I ran the two miles home from school in just 16 minutes, but I got driven to school along the same route at a speed of 30 miles per hour. What was my average speed, in miles per hour, for the round trip?

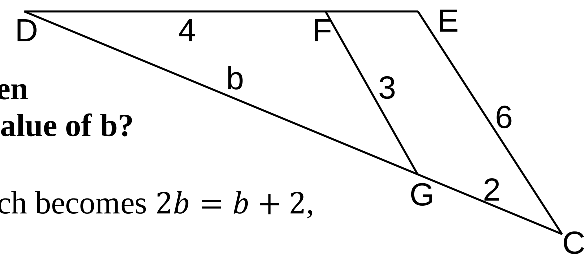
I traveled a total of $2 + 2 = 4$ miles, and it took a total of $16 + \frac{2}{30} \times 60 = 16 + 4 = 20$ minutes, so the overall average speed was 12 miles per hour.

24. What are the coordinates, in the form (x, y), when the point $(7, 9)$ is rotated $90°$ clockwise about the point $(-6, 2)$?

(7,9) is $7 - (-6) = 13$ units to the right and $9 - 2 = 7$ units above $(-6,2)$. After rotating 90° clockwise, the new point will be 13 units below and 7 units to the right of $(-6,2)$, which will be $(1, -11)$.

25. A right triangle with a $30°$ angle has a hypotenuse measuring 12 m. What is its area, in square meters?

If the hypotenuse is 12, the short side is $\frac{12}{2} = 6$, and the long leg is $6\sqrt{3}$, so the area is $\frac{1}{2} \cdot 6 \cdot 6\sqrt{3} = 18\sqrt{3}$.



26. In the ΔCDE to the right, $\overline{FG}||\overline{CE}$ and all given segment lengths are in meters. What is the value of b?

Similar triangles allow us to write $\frac{b}{3} = \frac{b+2}{6}$, which becomes $2b = b + 2$, giving $b = 2$.

27. What is the name for a triangle cevian which meets the opposite side at its midpoint?

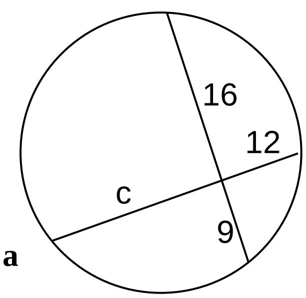
You have to have memorized that it's "median".

28. A pentagon has sides measuring 20 m, 1 m, 2 m, 3 m, and *x* m. What is the sum of the largest possible integer value of *x* and the smallest possible integer value of *x*?

The triangle inequality can be extended to other shapes; if the 1, 2, and 3 are all adding to the 20, the other side might be as long as 25 (so the shape has some area). Similarly, if they are all subtracting from the 20, the other side could be as small as 15, for an answer of $15 + 25 = 40$.

29. What is the area, in square meters, of a sector of a circle with a radius of 2 m and an arclength of 4 m?

The central angle of this sector will be $\frac{4}{2} = 2$ radians, so the area will be $\frac{2}{2\pi} \cdot \pi r^2 = r^2 = 2^2 = 4$.



30. Two chords intersect in the circle to the right, with all segment lengths given in meters. What is the value of *c*?

We can write $12 \cdot c = 16 \cdot 9 = 144$, so that $c = 12$.

31. What is the circumference, in meters, of a circle inscribed inside a regular hexagon with an area of $12\sqrt{3}$ m^2?

A hexagon is six equilateral triangles, each of which has an area of $2\sqrt{3}$ in this case. We can write $\frac{s^2\sqrt{3}}{4} = 2\sqrt{3}$, which becomes $s^2 = 8$, giving $s = \sqrt{8} = 2\sqrt{2}$. The radius of the inscribed circle will be the altitude of one of these triangles, which will be $\sqrt{3} \cdot \sqrt{2} = \sqrt{6}$, so that the circumference will be $2\pi\sqrt{6}$.

32. What is the name for a solid with nine planar faces?

We're expecting a lot of "nonahedron", which is what we initially thought would be the only answer, but it turns out that "enneahedron" is the preferred name. We're accepting both.

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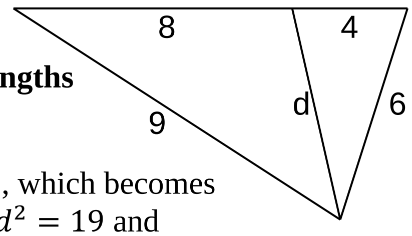

33. In the triangle to the right with one cevian, all segment lengths are given in meters. What is the value of *d*?

Stewart's Theorem gives $8 \cdot 6^2 + 4 \cdot 9^2 = 12d^2 + 8 \cdot 4 \cdot 12$, which becomes $4 \cdot 6 + 3 \cdot 9 = d^2 + 8 \cdot 4$, then $24 + 27 = d^2 + 32$, giving $d^2 = 19$ and $d = \sqrt{19}$.

34. What is the largest number of spheres you can have such that each sphere touches every other sphere?

You can get three spheres touching one another in a triangular pattern. Then you can add a fourth in a tetrahedral position. Finally, you can add a smaller one in their center, touching all three (or you could do a giant one that surrounds the other four), so the answer is 5.

35. Two concentric circles have radii that differ by 10 m. If the area between the two circles is 100π m^2, what is the length of a chord of the outer circle that is tangent to the inner circle?

The "differ by 10" is a red herring; this is the standard chord between circles problem, so the answer is $2 \cdot \sqrt{100} = 2 \cdot 10 = 20$.

36. What are the coordinates, in the form (x, y), of the center of the conic section $4x^2 + 5y^2 + 24x - 40y = 0$?

Completing the square gives $4(x+3)^2 + 5(y-4)^2 = 4 \cdot 3^2 + 5 \cdot 4^2$, so the center is $(-3,4)$.

37. What value(s) of *w* satisfy $3^{2w+1} - 28 \cdot 3^w + 9 = 0$?

This can be written as $3 \cdot (3^w)^2 - 28 \cdot 3^w + 9 = 0$, which is a quadratic in 3^w and thus factors to $(3 \cdot 3^w - 1)(3^w - 9) = 0$, so that $3^w = \frac{1}{3}$ or $3^w = 9$, and thus $w = -1$ or 2.

38. If $v(u) = 3u^2(u+1)$ and $t(s) = \frac{s}{s^2+1}$, what is the value of $v\big(t(-2)\big)$?

$$v\big(t(-2)\big) = v\left(-\frac{2}{5}\right) = 3\left(-\frac{2}{5}\right)^2\left(-\frac{2}{5}+1\right) = 3 \cdot \frac{4}{25} \cdot \frac{3}{5} = \frac{36}{125}$$

39. The half-life of Cherium is twelve minutes. How many grams of a 9000 kg sample of Cherium will remain after an hour?

60 minutes is five half-lives, so there will be $\frac{9000000}{2^5} = 90 \cdot 5^5 = 281{,}250$ grams left.

40. Express the base ten numeral 624_{10} as a base eleven numeral.

In base 11, the rightmost digits represent 1's, 11's, & 121's. We'll need five 121's, for a total of 605, leaving 19 for the other digits. That's one 11 and eight 1's, for an answer of 518_{11}.

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41. Express the difference $4826_9 - 1356_9$ as a base nine numeral.

Subtracting in a base other than 10 works the same, except that carrying is slightly different. $6 - 6$ is still 0, $2 - 5$ becomes $11 - 5 = 6$ through borrowing, leading to $7 - 3 = 4$, followed by $4 - 1 = 3$, for an answer of 3460.

42. What is the largest number less than 100 that leaves a remainder of 1 when it is divided by three and a remainder of 3 when it is divided by 4?

Numbers like this will be $LCM(3,4) = 12$ apart, so if we find one, we can find the rest. Those that leave a remainder of 3 when divided by 4 are 3, 7, aha! So we need the largest number less than 100 that is 7 more than a multiple of 12. $96 + 7$ will be too high, so we want $84 + 7 = 91$.

43. What is the 268th term of an arithmetic sequence with first term 417 and common difference 34?

$417 + 267 \cdot 34 = 417 + 9078 = 9495$

44. Evaluate: $\prod_{q=1}^{10} \frac{q+1}{q+3}$

This will be $\frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} \cdot \frac{5}{7} \cdot \frac{6}{8} \cdot \frac{7}{9} \cdot \frac{8}{10} \cdot \frac{9}{11} \cdot \frac{10}{12} \cdot \frac{11}{13}$, which allows a LOT of canceling, leaving $\frac{2 \cdot 3}{12 \cdot 13} = \frac{1}{2 \cdot 13} = \frac{1}{26}$.

45. What is the sum of the first 9 terms of a geometric sequence with first term 120 and common difference ratio $\frac{1}{2}$?

$S_n = \frac{a(1-r^n)}{1-r} = \frac{120\left(1-\left(\frac{1}{2}\right)^9\right)}{1-\frac{1}{2}} = \frac{240(2^9-1)}{2^9} = 15 \cdot \frac{511}{2^5} = \frac{7665}{32}$

46. What is the mode of the data set {1, 2, 6, 7, 8, 0, 4, 1, 3, 8, 7, 8, 0, 1, 7, 8, 0}?

There are four 8's, and at most 3 of anything else, making the answer 8.

47. When two cards are drawn from a standard 52-card deck, what is the probability that the first one has a lower rank than the second one?

The probability that they match is $\frac{52}{52} \cdot \frac{3}{51} = \frac{1}{17}$, so the probability that they do not is $1 - \frac{1}{17} = \frac{16}{17}$. Half the time they don't match, the first one will be lower than the second, for an answer of $\frac{1}{2} \cdot \frac{16}{17} = \frac{8}{17}$.

48. When three fair six-sided dice are rolled, what is the probability that exactly two of them show the same number?

There are $6^3 = 216$ ways to rolls three dice. We're looking for ABB in some order, so there are 6 choices for A and 5 choices for B, as well as three orders they could roll in, for a probability of $\frac{6\cdot5\cdot3}{216} = \frac{15}{36} = \frac{5}{12}$.

49. What is the shortest distance from the point $(-4, 1, -3)$ to the plane $3x - 4y - 5z = 6$?

Similar to the 2D version, the distance will be $\frac{|3(-4)-4\cdot1-5(-3)-6|}{\sqrt{3\hat{}2+(-4)^2+(-5)^2}} = \frac{|-12-4+15-6|}{\sqrt{9+16+25}} = \frac{|-7|}{\sqrt{50}} = \frac{7}{5\sqrt{2}} = \frac{7\sqrt{2}}{10}$.

50. How many subsets of {7, 45, 8, 9, 14, 5, 79} are supersets of $(8, 45, 9\}$?

8, 45, and 9 must be in the set, and each of 7, 14, 5, and 79 may choose whether or not to join (two choices each), for an answer of $2^4 = 16$.

51. Express the spherical coordinates $\left(4, \frac{3\pi}{2}, \frac{5\pi}{6}\right)$ as polar coordinates. The spherical coordinates list the radius, azimuthal angle, and polar angle in that order.

The azimuthal angle will remain the same, while the radius and polar angle will produce a new radius in the x-y plane and a z-value. $z = \rho\cos\phi = 4\cos\frac{5\pi}{6} = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$, and $r = \rho\sin\phi = 4\sin\frac{5\pi}{6} = 4\left(\frac{1}{2}\right) = 2$, for an answer of $\left(2, \frac{3\pi}{2}, -2\sqrt{3}\right)$.

52. Evaluate: $\cot\left(-\frac{57\pi}{6}\right)$

First, we can get rid of any number of $2\pi = \frac{12\pi}{6}$, leaving us with $\frac{9\pi}{6} = \frac{3\pi}{2}$. Cotangent is $\frac{adjacent}{opposite}$, and on the y-axis the adjacent side of our reference triangle is 0, so our answer is 0.

53. Evaluate: $\lim_{n\to4}\frac{3n^3-14n^2+9n-4}{4-n}$

Substituting gives $\frac{3\cdot4^3-14\cdot4^2+9\cdot4-4}{4-4} = \frac{192-224+36-4}{0} = \frac{228-228}{0} = \frac{0}{0}$, which is indeterminate, so we'll have to do some algebraic manipulation and try again. Factoring out $(n-4)$ is the obvious route, giving $\frac{(n-4)(3n^2-2n+1)}{4-n} = -3n^2 + 2n - 1$. Substituting now gives $-3\cdot4^2 + 2\cdot4 - 1 = -48 + 8 - 1 = -41$.

54. Evaluate: $\lim_{n\to e}\frac{e^n-e^e}{n-e}$

This is the definition of the derivative when $f(x) = e^x$, at the point $x = e$. $f'(x) = e^x$, so $f'(e) = e^e$.

55. What are the coordinates, in the form (x, y) of the leftmost critical point of $y = 2x^3 - 3x^2 - 12x - 5$?

The critical points are where $\frac{dy}{dx} = 0$, so let's take a derivative. $\frac{dy}{dx} = 6x^2 - 6x - 12 = 0$ becomes $x^2 - x - 2 = 0$, which factors to $(x-2)(x+1) = 0$ with roots of 2 and -1. -1 is the leftmost of these, and the corresponding y-value is $y = 2(-1)^3 - 3(-1)^2 - 12(-1) - 5 = -2 - 3 + 12 - 5 = 12 - 10 = 2$, for an answer of $(-1,2)$.

Harder Problems

56. Evaluate: 57.9×34.16

The standard algorithm gives 1977.864. This is essentially non-decimal multiplication, with a decimal point added at the end, $2 + 1 = 3$ digits from the end.

57. What percent of 32 is 76?

$\frac{76}{32} \times 100 = \frac{19}{8} \times 100 = \frac{19}{2} \times 25 = \frac{475}{2}$

58. Evaluate: $121^3 - 119^3$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2) = (a-b)[(a-b)^2 + 3ab]$, which in this case means $2[4 + 3 \cdot 121 \cdot 119] = 2[4 + 3(120^2 - 1^2)] = 2(4 + 3 \cdot 14399) = 2 \cdot 43201 = 86{,}402$.

59. What is the solution, in the form (j, k, m), of the system of equations $j + k + m = -6$, $j - k - 2m = 4$, and $-3j + 2k + m = 7$?

Adding the first two equations gives $2j - m = -2$. Adding twice the second equation to the third gives $-j - 3m = 15$. Adding twice this to the first generated equation gives $-7m = 28$, so $m = -4$. Working backwards, $-j + 12 = 15$, so $-j = 3$, and $j = -3$. Finally, $-3 + k + (-4) = -6$ gives $k = 1$ for an answer of $(-3,1,-4)$.

60. Jack could build the brick wall in twelve hours and Jill could build it in ten hours. They're asked to work on the wall together, but because they talk to each other, they lay 100 fewer total bricks per hour than they would have if they were working separately, and thus it takes them seven hours to complete the wall. How many bricks were in the wall, to the nearest brick?

Together, they work at a speed of $\frac{1}{12}+\frac{1}{10}=\frac{11}{60}$ walls per hour. If there are B bricks in a wall, then their combined speed is $\frac{11}{60}B$, except that they're chatty, so it's really $\frac{11}{60}B-100$. Working at this speed for 7 hours, they build a whole wall, which is B bricks, so we can write $7\left(\frac{11}{60}B-100\right)=B$. This becomes $\frac{77}{60}B-700=B$, then $\frac{17}{60}B=700$, giving $B=700\cdot\frac{60}{17}=\frac{42000}{17}\cong 2470.6\cong 2471$.

61. The IB group decided to order the World's Best Pizza and split the cost evenly. If there had been one more member, each person would have paid $.40 less. If there had been one fewer member, each person would have paid $.50 more. How many people are in the IB group?

We can write $C=np=(n+1)(p-.4)=(n-1)(p+.5)$, where C is the cost of the pizza, n is the number of people in the group, and p is the amount paid per person. Expanding the two $np=()()$ equations gives $.4=-.4n+p$ and $.5=.5n-p$. Adding these equations gives $.9=.1n$, so that $n=9$.

62. What is a solution, in the form (u,v,w), of the system of equations $u+v+w=12$, $uv+vw+uw=-43$, and $uvw=30$?

These variables could be the roots of the polynomial $t^3-12t^2-43t-30=0$. One root is probably large and positive so that t^3 can compensate for the three negative terms, and of course it needs to be a factor of 30, so we'll try 10. No, 1000 is already dwarfed by -1200. Let's try 15: $3375-2700-645-30=3375-3375=0$. Yay! Once we know this root, we can factor to get $(t-15)(t^2+3t+2)=(t-15)(t+1)(t+2)=0$ with roots of 15, -1, and -2, for many possible orders of $(-1,-2,15)$.

63. Xerxes was 12 when Yolanda was twice Zed's age, and Zed was 3 when Yolanda was twice Xerxes' age. If they all share the same birthday in different years, how old will Yolanda be when Zed is 30?

There are three years to consider; in the first, their ages were 12, $2z$, and z. In the second, which was $z-3$ years earlier, their ages were $12-(z-3)=15-z$, $2(15-z)=2z-(z-3)$, and 3. Yolanda's age that year was calculated two different ways, and leads to $30-2z=z+3$, then $3z=27$, giving $z=9$. That means that in the first year, their ages were 12, 18, and 9, and when Zed is thirty in 21 years, Yolanda will be $18+21=39$.

64. A jaguar at position $(5,3)$ wishes to drink from the stream $(x-y=8)$, then return to the tree where she stored a gazelle's body at position $(-2,1)$. What is the shortest distance she can travel?

Consider point P on the river which corresponds to the shortest total distance. If point J were reflected across the river to J', path JP would be the same length as J'P, so path JPT would be the same length as J'PT, and this would be the shortest such path, which should be a straight line now that J' and T are on opposite sides of the river. J' is $(11, -3)$, so the length of the segment J'T through P is $\sqrt{\left(11-(-2)\right)^2+(-3-1)^2}=\sqrt{13^2+4^2}=\sqrt{169+16}=\sqrt{185}$.

65. A professor computes the average of her students' scores on a recent test, getting a value of 70. However, she realized that although she had divided by the correct number of student scores, she had forgotten to include one test score when she computed the total of the scores. She adds the missing score, recalculating the total correctly, but absent-mindedly also adds one to the number of scores, getting a new incorrect average of 71. Realizing she's made another oversight, she correctly calculates the average to be 72. What is the lowest possible value of the missing test score that started all this madness?

We can write the equations $70=\frac{\Sigma-X}{n}$, $71=\frac{\Sigma}{n+1}$, and $72=\frac{\Sigma}{n}$, where Σ is the sum of all the scores, X is the initially-missing score, and n is the number of scores. Cross-multiplying gives $70n=\Sigma-X$, $71n+71=\Sigma$, and $72n=\Sigma$. Subtracting the first from the third gives $X=2n$. Subtracting the second from the third gives $n=71$, making $X=2n=2\cdot 71=142$.

66. A right triangle has an area of 84 m^2 and a perimeter of 56 m. What is the length, in meters, of its hypotenuse?

Because the area and perimeter are both integers, it seems likely we're looking for a Pythagorean triple. 3-4-5 is the smallest, with a perimeter of 12, but this isn't a factor of 56 so we're probably not looking for one if its multiples. 5-12-13 is next with a perimeter of 30, but this is also unlikely. 7-24-25 has a perimeter of 56, and an area of 84, so this is our solution, giving an answer of 25.

67. What is the smallest possible perimeter, in meters, of a rectangle with an area of 1485 m^2 and integer side lengths when measured in meters?

$1485=5\cdot 297=5\cdot 3^2\cdot 33=3^3\cdot 5\cdot 11$, and we'd like side lengths that are close to one another to minimize the perimeter. 27 & 55 pops out, then 45 & 33 (better). Focusing on the 11, one of the sides will have to be 11, 22 (not possible), 33, 44 (not possible), 55, etc., so our solution of 45 & 33 will be the smallest perimeter, which turns out to be $2(45+33)=2\cdot 78=156$.

68. Two circles have radii of 41 m and 18 m, and have their centers 295 m apart. What is the length in meters of one of their common internal tangents?

Drawing the circles, the segment between their centers, a tangent, and the radii to the tangent gets us most of the way to a solution. The radii are perpendicular to the tangent, and if we extend one radius the length of the other, a copy of the tangent will now reach from this extended radius to the other center, forming a rectangle. The copy of the tangent is a leg of a right triangle with hypotenuse 295 and other leg (extended radius) of $41 + 18 = 59$, so that our answer will be $\sqrt{295^2 - 59^2}$. This doesn't look fun, but if you notice that $295 = 5 \cdot 59$, we can jump to $59\sqrt{5^2 - 1^1} = 59\sqrt{24} = 118\sqrt{6}$.

69. On a tessellated plane, every vertex is surrounded by a combination of squares and equilateral triangles. If no two squares share a side, what fraction of the plane is covered by squares? Note: only one size of square and one size of equilateral triangle are used.

Any vertex will need to be surrounded by 2 squares and three triangles. Any square will need to be surrounded by four triangles. When two triangles meet face-to-face, they'll be surrounded by squares. Following these rules, you can create a semi-regular tessellation that is the second one pictured at gwydir.demon.co.uk/jo/tess/sqtri.htm. It's a little hard to think about the ratio of squares to triangles in this figure, especially if you only draw a little bit of it. Gwydir.demon.co.uk/jo/tess/grids.htm shows some shadings of small parts of this grid that demonstrate that the ratio is 1 square for every 2 triangles. A shading that is not shown but which we think is simplest is to shade a square, all four adjoining triangles, and one other square sharing a face with one of those triangles. This shape generates a regular tessellation.

Anyhow, all of that gets us to a ratio of 1 square to 2 triangles, so the fraction of the plane covered by squares is $\frac{1}{1+2\cdot\frac{\sqrt{3}}{4}} = \frac{1}{\frac{4+2\sqrt{3}}{4}} = \frac{4}{4+2\sqrt{3}} = \frac{16-8\sqrt{3}}{4} = 4 - 2\sqrt{3}$.

70. What is the first time after 3:15:00 AM that the hour and minute hands of a standard 12-hour analog clock form the same angle as they did at 3:15? Express your answer to the nearest second.

At 3:15, the minute hand is just behind the hour hand, and a few minutes later it will be just ahead. Specifically, the angle between them at 3:15 will be $90 - 15 \cdot \frac{11}{2} = \frac{180-165}{2} = \frac{15}{2}$ degrees, and the number of minutes until the other arrangement will be $2 \cdot \frac{15}{2} \div \frac{11}{2} = \frac{30}{11}$. This is $2\frac{8}{11}$ minutes, which is 2 minutes and $\frac{8}{11} \cdot 60 = \frac{480}{11} = 43\frac{7}{11}$ seconds, for an answer of 3:17:44.

71. Line segments are drawn from a triangle's incenter to each of its vertices, dividing it into areas of 10 m^2, 17 m^2, and 21 m^2. What is the perimeter, in meters, of the triangle?

All three of these triangular sub-areas have the same height (the inradius), so their bases (the sides of the original triangle) must be in the same proportion as their areas, so we can call the sides of the original triangle $10b$, $17b$, and $21b$. Using Heron's Formula, $s = 24b$, giving an area of $\sqrt{24b \cdot 14b \cdot 7b \cdot 3b} = b^2 \cdot 3 \cdot 7 \cdot 4 = 84b^2$. We also know the area is $10 + 17 + 21 = 48$, so $b^2 = \frac{48}{84} = \frac{4}{7}$, so $b = \sqrt{\frac{4}{7}} = \frac{2}{\sqrt{7}} = \frac{2\sqrt{7}}{7}$ and the perimeter is $48 \cdot \frac{2\sqrt{7}}{7} = \frac{96\sqrt{7}}{7}$.

72. At how many points do the graphs of $y = \log_3 \frac{x}{4}$ and $\left(x - \frac{7}{2}\right)^2 + \left(y + \frac{5}{2}\right)^2 = 9$ intersect?

The log graph shoots up from $-\infty$ for small positive x's, passes through $(4,0)$, and heads to the right, increasing slowly. The circle has a center at $\left(\frac{7}{2}, -\frac{5}{2}\right)$ and a radius of 9. The circle's center is almost directly below the log's x-intercept, and probably contains it, so they'll intersect in two points (the log goes into the circle, the log leaves the circle).

73. At the pizza buffet, Sam can eat two slices of pizza every minute until there are no more slices available, at which point he leaves, even if another pizza appears at the instant he is leaving. The restaurant always slices their pizzas into eight slices, and brings out a whole pizza every *M* minutes. Strangely, there are never any other diners when Sam is at the buffet. Two weeks ago, there were four pizzas on the buffet at the moment he arrived, and he left after 20 minutes. Last week, there were six pizzas on the buffet at the moment he arrived, and he left after 36 minutes. What is the sum of all possible integer values of *M*?

Two weeks ago, exactly one pizza must have come out during his 20 minutes there. Specifically, two or more did NOT come out, so they must come less often than every 10 minutes. Actually, 10 could work, as one of the four might have arrived the same instant Sam did, another at the 10-minute mark, and a third as Sam was leaving at 20 minutes. So, $M \geq 10$. One week ago, exactly three pizzas must have come out in 36 minutes. Specifically, it wasn't just two pizzas, so they must come more frequently than every 18 minutes, and it wasn't four pizzas, so they must come less frequently than every 9 minutes. 9 doesn't work because of our $M \geq 10$ constraint; could 18 work? If one arrived one second after Sam, others would have arrived at 18+ minutes and 36+ minutes, which doesn't work. So, $10 \leq M \leq 17$, which means our answer is $10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 = \frac{8}{4} \cdot 27 = 108$.

74. If $\log_2 3 = n$ and $\log_5 2 = p$, express $\log 9$ in terms of n and p.

$$\log 9 = \log_{10} 9 = 2\log_{10} 3 = \frac{2}{\log_3 10} = \frac{2}{\log_3 2 + \log_3 5} = \frac{2}{\frac{1}{\log_2 3} + \log_3 2 \cdot \log_2 5} = \frac{2}{\frac{1}{n} + \frac{1}{n} \cdot \frac{1}{p}} = \frac{2}{\frac{p+1}{np}} = \frac{2np}{p+1}$$

75. What is the maximum value of the expression $|g - h| + |g - j| + |k - g|$ if $\leq k \leq j \leq h \leq g \leq 10$?

Given that we know $g \geq h$and other facts, this becomes $(g - h) + (g - j) + (g - k) = 3g - (h + j + k)$. g can be 10, and h, j, and k can be 0, for an answer of 30.

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Solutions

76. What is the sum of the reciprocals of the squares of the roots of $2r^3 - 5r^2 - 3r + 4 = 0$?

If the roots are a, b, and c, we're looking for $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{b^2c^2 + a^2c^2 + a^2b^2}{a^2b^2c^2}$. We know that $a + b + c = -\frac{-5}{2} = \frac{5}{2}$, $ab + bc + ac = -\frac{3}{2}$, and $abc = -\frac{4}{2} = -2$, so the denominator is just $(-2)^2 = 4$. The numerator looks a bit like $(ab + bc + ac)^2$, but there will be extra terms to take care of. $(ab + bc + ac)^2 = a^2b^2 + b^2c^2 + a^2c^2 + 2ab^2c + 2a^2bc + 2abc^2$. The first part is the numerator, and the second part is $2(abc)(a + b + c)$. Our answer is therefore $\frac{\left(-\frac{3}{2}\right)^2 - 2(-2)\cdot\frac{5}{2}}{4} = \frac{\frac{9}{4} + 10}{4} = \frac{49}{16}$.

77. What is the minimum value of the expression $-\left\lfloor\frac{x}{2}\right\rfloor + \left\lfloor\frac{x}{3}\right\rfloor + \left\lfloor\frac{x}{5}\right\rfloor - \left\lfloor\frac{x}{7}\right\rfloor$ for $0 \le x \le 100$?

Generally, this will grow like $\frac{x}{3} + \frac{x}{5} - \left(\frac{x}{2} + \frac{x}{7}\right) = \frac{8x}{15} - \frac{9x}{14}$, so that it will be a negative number that gets more negative the larger x is. To get its smallest value, we should use our largest x. However, there are many jump discontinuities to consider. Specifically, if x is a multiple of 2 or 7, that's good, as we're subtracting an optimally large number. Similarly, if x is a multiple of 3 or 5, that's bad, as we're adding an optimally large number. Ideally, we'd have a multiple of 2 & 7 near 100 that wasn't a multiple of 3 or 5, and there is one: 98. Plugging this in gives $-49 + 32 + 19 - 14 = -63 + 51 = -12$.

78. How many positive integers are factors of 4752 and multiples of 18?

$18 = 2 \cdot 3^2$ and $4752 = 2^2 \cdot 1188 = 2^4 \cdot 297 = 2^4 \cdot 3^2 \cdot 33 = 2^4 \cdot 3^3 \cdot 11$. The numbers we're seeking can have factors from 2^1 to 2^4 (four choices), from 3^2 to 3^3 (two choices), and from 11^0 to 11^1 (two choices), for an answer of $4 \cdot 2 \cdot 2 = 16$.

79. The floor of a room is in the shape of a parallelogram with sides measuring 3 m and 6 m and corner angles of $120°$ and $60°$. The room is tiled with a tessellation of equilateral triangles that each measure 50 cm on each side. As part of a remodel, a contractor draws the longer diagonal of this parallelogram on the floor. How many tiles does that line intersect? Note: intersecting a tile ONLY at a vertex does NOT count as intersecting the tile.

A rough sketch might reveal that the room is built of four 3 m triangles. Overall, it is divided into 6 columns, each containing 24 triangles. Thus, as the diagonal traverses one diagonal, it will move up 4 triangles, passing through a vertex at this point (this shape is similar to the entire room). It will do this 6 times, for an answer of $6 \cdot 4 = 24$.

80. Positive integers are all written in a line. Once this is done, the number 672 is colored blue, as is every 150th number afterward (833, 983, etc.). Once this is done, the number 264 is colored red, as is every 480th number afterward (744, 1224, etc.). What is the fewest numbers that appear between a red number and a blue number?

The greatest common factor of 150 and 480 is 30, so this is the minimum amount by which the difference between two differently-colored numbers can be adjusted. The initial numbers, 672 and 264, differ by $672 - 264 = 408$. Adjusting by 390 gives $408 - 390 = 18$, but adjusting again gives $18 - 30 = -12$, which is really 12 with the order of blue & red reversed. There will be $12 - 1 = 11$ uncolored numbers between two such numbers.

81. How many seven-digit palindromes are even and contain at most two different digits?

The numbers must be of the form ABCDCBA, where A is even and we're using at most two digits. Our numbers could be of the form AAAAAAA (1 way), ABAAABA (3 ways to place B), ABBABBA (3 ways to place B's), and ABBBBBA (1 way). In the first case there are 4 choices for A, so $4 \cdot 1 = 4$ numbers total. For the other cases, there are 4 choices for A and 9 choices for B, so there are $4 \cdot 9(3 + 3 + 1) = 36 \cdot 7 = 252$, for an answer of $252 + 4 = 256$. Was there an alternate way to compute this as 2^8, or is this a coincidence?

82. What is the twentieth term of the sequence beginning 8, 12, 19, 29, 42, …?

The differences are 4, 7, 10, 13, …, which is an arithmetic sequence. The 20th term of the original sequence will be 8 plus 19 terms of the difference sequence. The 19th term of the difference sequence will be $4 + 18 \cdot 3 = 4 + 54 = 58$, so the sum of those 19 terms will be $\frac{19}{2} \cdot (4 + 58) = \frac{19 \cdot 62}{2} = 19 \cdot 31 = 589$, making our answer $8 + 589 = 597$.

83. What is the next term of the harmonic sequence 630, 504, 420, …?

A harmonic sequence is a constant divided by the terms of an arithmetic sequence, so that constant can be expressed as $630n = 504(n + d) = 420(n + 2d)$. $630 = 2 \cdot 3^2 \cdot 5 \cdot 7$, $504 = 2^3 \cdot 3^2 \cdot 7$, and $420 = 2^2 \cdot 3 \cdot 5 \cdot 7$, so their greatest common factor is $2 \cdot 3 \cdot 7 = 42$. $630 = 42 \cdot 15$, $504 = 42 \cdot 12$, $420 = 42 \cdot 10$. The LCM of 15, 12, and 10 is 120, so the constant could be $42 \cdot 120$, and the terms could be divided by 8, 10, and 12. The next term would then be divided by 14, making it $42 \cdot \frac{120}{14} = 3 \cdot 120 = 360$.

84. What is the missing term of the sequence 542, 582, 662, ___, 818, 882, 1010, …?

The differences are 40, 80, ?, ?, 64, and 128. It's hard to see, but hopefully somebody stumbles on the fact that $5 \cdot 4 \cdot 2 = 40$, $5 \cdot 8 \cdot 2 = 80$, etc. That would make the third difference $6 \cdot 6 \cdot 2 = 72$, so that the missing term would be $662 + 72 = 734$. If so, the next difference would be $7 \cdot 3 \cdot 4 = 84$, making the 818 term be $734 + 84 = 818$, which works! So, our answer is 734.

85. A sequence has first term $m_1 = 11$ and subsequent terms $m_n = 3m_{n-1} - 3^n$. What is the fifth term of this sequence?

$m_2 = 3 \cdot 11 - 3^2 = 33 - 9 = 24$, $m_3 = 3 \cdot 24 - 3^3 = 72 - 27 = 45$,

$m_4 = 3 \cdot 45 - 3^4 = 135 - 81 = 54$, and $m_5 = 3 \cdot 54 - 3^5 = 162 - 243 = -81$.

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Solutions

86. A point is chosen randomly on a yardstick, and the yardstick is broken at that point. Then a point is chosen randomly on the larger piece, and that piece is broken at that point. What is the probability that the shortest of the three pieces is shorter than 6 inches?

There is a $\frac{6+6}{36} = \frac{12}{36} = \frac{1}{3}$ chance that the first point chosen produces a piece shorter than 6 inches, and a $1 - \frac{1}{3} = \frac{2}{3}$ chance that it does not. In the latter case, the length of the longer piece (call it x) could be anywhere from 24 to 18 (if it's less than 18, the other piece is the longer piece), and the point chosen could range from 0 to x. Within this space, any time our point is within 6 of either end of the stick, it produces our desired result, giving us a "good" probability of $\frac{12}{x}$, for a secondary probability of $\frac{\int_{18}^{24} \frac{12}{x} dx}{24-18} = \frac{12 \ln x|_{18}^{24}}{6} = 2(\ln 24 - \ln 18) = 2 \ln \frac{24}{18} = 2 \ln \frac{4}{3}$. This makes our overall answer $\frac{1}{3} + \frac{2}{3} \cdot 2 \ln \frac{4}{3} = \frac{1}{3} + \frac{4}{3} \ln \frac{4}{3}$.

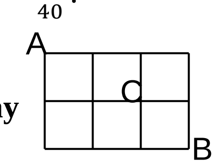
87. Two players take turns drawing (and keeping) a single marble from a bag that initially contains three green marbles and three white marbles. The winner is the first player to get two marbles that are the same color. What is the probability that the first person wins the game?

The first player wins the game if it goes XXX, XYX, XXYYZ, or XYYXZ. In all cases, the first X has probability 1 because we don't actually care what happens here, just whether we match it or not later. The probability of XXX is thus $\frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20}$ (I don't reduce on problems like this, to make later addition easier). The probability of XYX is $\frac{3}{5} \cdot \frac{2}{4} = \frac{6}{20}$, XXYYZ is $\frac{2}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{4}{20}$ (again, trying to simplify addition later), and XYYXZ is $\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{4}{20}$, for an answer of $\frac{2+6+4+4}{20} = \frac{16}{20} = \frac{4}{5}$.

88. In D&D 5e, a player with "advantage" rolls two 20-sided dice and uses the higher of the two numbers shown. What is the expected value of the higher of the two numbers?

There are $20 + 20 - 1 = 39$ ways to get a higher value of 20, $19 + 19 - 1 = 37$ ways to get a 19, $18 + 18 - 1 = 35$ ways to get an 18, etc. and there are $20 \cdot 20 = 400$ total ways to roll the two dice. The expected value will thus be $\frac{39}{400} \cdot 20 + \frac{37}{400} \cdot 19 + \cdots \frac{3}{400} \cdot 2 + \frac{1}{400} \cdot 1 = \frac{\sum_{n=1}^{20} n(2n-1)}{400} = \frac{\sum_{n=1}^{20} (2n^2 - n)}{400} = \frac{\frac{2(20 \cdot 21 \cdot 41)}{6} - \frac{20 \cdot 21}{2}}{400} = \frac{140 \cdot 41 - 210}{400} = \frac{5740 - 210}{400} = \frac{5530}{400} = \frac{553}{40}$.

89. In the grid of unit squares to the right, how many paths of length 7 are there from A to B that pass through C at least once? Note: it is not okay for a path to pass through vertices and/or segments multiple times.



Generally, the answer to the problem will be the number of ways to get from A to C multiplied by the number of ways from C to B. If you go directly from A to C, you will take a path of length 3, and one of your steps will be down, giving $3c1 = 3$ ways. In this case, you must take four steps from C to B without the path crossing itself or the path you took from A to C. It turns out there are only two paths of length 4 from C to B: URDD and LDRR. However, any arrival at C will negate one of these possibilities, so there is really just one long way from C to B once you've gone a short way from A to C, for a subtotal of $3 \cdot 1 = 3$. Similarly, there are two direct routes from C to B, and long routes from A to C can be RRRDL, RDDRU, DRURD, DRDRU, DDRUR, or DDRRU (6 ways). Anything ending in L or U (4 cases) only allows one path from C to B without overlapping, but the other two paths allow both direct C to B options, for a subtotal of $4 \cdot 1 + 2 \cdot 2 = 4 + 4 = 8$. Adding the subtotals gives an answer of $3 + 8 = 11$.

90. What is the area of a triangle with vertices at the points $(7, 9)$, $(-3, 5)$, and $(6, -5)$?

This triangle fits inside a rectangle with a width of $7 - (-3) = 10$ and a height of $9 - (-5) = 14$, with an area of $10 \cdot 14 = 140$, and is the remainder when three right triangles are cut from that rectangle. The right triangles have a combined area of $\frac{1}{2}\Big((9-5)(7-(-3)) + (9-(-5))(7-6) + (5-(-5))(6-(-3))\Big) = \frac{1}{2}(4 \cdot 10 + 14 \cdot 1 + 10 \cdot 9) = \frac{1}{2}(40 + 14 + 90) = \frac{1}{2}(144) = 72$, for an answer of $140 - 72 = 68$.

91. A set of seven integer test scores from 0 to 100 inclusive has a mean of 70, a unique mode of 80, and a range of 60. What is the smallest possible value of the median?

Arranging the scores in ascending order, they're X, __, __, Y, 80, 80, and $X + 60$. Because the mean is 70, the sum must be $7 \cdot 70 = 490$. For this fixed sum, we'd like a low value of Y. To accomplish this, we'd like the other elements to be as high as possible, so we can write them as $40, Y - 2, Y - 1, Y, 80, 80, 100$, then write $297 + 3Y = 490$, so that $3Y = 193$ and $Y = 64.x$, meaning that the answer is 65.

92. Set L is the set of positive four-digit integers that contain the digit 4, Set K is the set of positive three-digit integers that do not contain a 5, and Set J is the set of integers with at least two matching digits. How many elements are in the set $(J \cap K) \cup L$?

We want everything in L, and anything additional that is in both J & K. There are 9000 four-digit numbers, but $8 \cdot 8 \cdot 7 \cdot 6 = 64 \cdot 42 = 2688$ of them don't contain a four, so L has 6312 elements. J&K will be three-digit numbers without a five, but with at least two matching digits. Such numbers could be of the form AAA, ABA, ABB, or AAB. In the first case, there are 8 such numbers. In the three later cases, there are $8 \cdot 8 = 64$ such numbers, so that $J \cap K$ has $8 + 3 \cdot 64 = 8 + 192 = 200$ elements. Because the latter are all three-digit numbers, and L is all four-digit numbers, there is no overlap, so the union is $6312 + 200 = 6512$.

93. In the cryptarithm below, each instance of a letter represents the same digit (0-9), and different letters represent different digits (e.g. if one A is a 1, all A's are 1's and B's cannot be 1's). What is the smallest possible value of the six-digit number ABCDEF?

$$\begin{array}{r} AB \\ \times\, CD \\ \hline BBB \\ \underline{DDDE} \\ AFEB \end{array}$$

There are often many ways to approach cryptarithms, but in this case the numbers BBB and DDD jumped out; they're both multiples of $111 = 3 \cdot 37$. This means that AB must be 37 or 74, and that C & D must each be 3, 6, or 9. $3 \cdot 37$ is not allowed (two 3's), $6 \cdot 37 = 222$ might be DDD (and E is obviously 0), $9 \cdot 37 = 333$ is not allowed (two 3's again), $3 \cdot 74 = 222$ might be DDD again, $6 \cdot 74 = 444$ might be BBB, and $9 \cdot 74 = 666$ might be DDD. There's only one option for BBB, which is 444, so the problem must be $74 \times C6$, where C is either 3 or 9. Because the final product is in the ball park of 100 times AB, the leading 9 seems likely, and following through gives $74 \cdot 96 = 444 + 6660 = 7104$, making the answer 749601.

94. An ant is on the face of a cube with edges measuring 6 m. The ant is currently at Point A, which is 1 m from one edge and 2 m from another edge. The ant wishes to travel to Point B, which is also 1 m from one edge and 2 m from another edge (there are many possible locations of this point). The ant will walk the shortest distance possible on the exterior of the cube to accomplish this. What is the longest distance, in meters, the ant could have to travel?

Consider the ant's starting point to be on the top of the cube, near the southern left vertex. Now consider unfolding the cube, so that the four sides are each in the same plane as the top. The opposite vertex appears twice in the upper right of this figure (once on the right and north faces), and clearly this is the area that is furthest from the ant. The bottom face, when unfolded into this plane, could extend either the right or north faces, offering locations near the opposite vertex that are even further from the ant. There are two possible starting points on the top and two possible ending points on the bottom (the problem is symmetric, so it is irrelevant which side face you traverse), but you can show that the shortest path between the farthest pair is $\sqrt{(5+6+2)^2+(6-1-2)^2} = \sqrt{13^2+3^2} = \sqrt{169+9} = \sqrt{178}$.

95. When five student line up for recess, Katie is exactly two places ahead of Isabella, Jesus is behind Li, Henri and Isabella are immediately adjacent (in either order), and Li is either first or last in line. If Jesus is behind Henri, write the first letters of each person's name in order from the front of the line to the back of the line.

When solving problems like this, I tend to take notes like KXI, L-J, HI/IH, SL/LE, and H-J. Then I start combining things that are obvious, such as SL-H-J, which then leads to LKHIJ. On scratch paper I would have done this vertically instead of horizontally, so it would be easier to think about "ahead of", etc.

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Solutions

96. A triangle has sides measuring 18 m and 24 m, with a $120°$ angle between them. What is the length, in meters, of the third side?

The Law of Cosines gives
$\sqrt{18^2+24^2-2\cdot 18\cdot 24\cdot\cos 120} = 6\sqrt{3^2+4^2-2\cdot 3\cdot 4\left(-\frac{1}{2}\right)} = 6\sqrt{9+16+12} =$
$6\sqrt{37}$.

97. What is the area, in square meters, of a triangle with sides measuring 5 m, $3\sqrt{5}$ m, and $2\sqrt{10}$ m?

You could use Heron's Formula here, but it would get messy pretty quickly. If you realize this might be a problem like #90, it can be a lot easier. The sides are $\sqrt{25}$, $\sqrt{45}$, and $\sqrt{40}$. $25=5^2+0^2=3^2+4^2$, $45=3^2+6^2$, and $40=2^2+6^2$ Noticing that $6+0=6$ and $3+2=5$, this triangle should fit in a 5x6 box! Yes, using the origin, $(6,2)$, and $(0,5)$ produces the desired lengths, and gives an area of $5\cdot 6-\frac{1}{2}(6\cdot 3+6\cdot 2)=30-\frac{1}{2}(18+12)=30-\frac{30}{2}=30-15=15$.

98. In how many points do $y = 9\cos 8\pi x$ and $y = 10 - e^{11x}$ intersect?

The first function oscillates between 9 and -9 with a period of $\frac{2\pi}{8\pi}=\frac{1}{4}$. It passes through $(0{,}9)$, $\left(\frac{1}{16},0\right)$, $\left(\frac{1}{8},-9\right)$, etc. The second function is basically exponential, but steeper, flipped vertically, and offset by 10, so near $x=-\infty$ it has a value just below 10, and for most positive values of x it's extremely negative. The only easy point to determine is (0.9), so that's one point of intersection so far. They will never intersect to the left of this, but we need to figure out where the exponential crosses $y=-9$ to see how many times they intersect to the right of this. $-9=10-e^{11x}$ becomes $e^{11x}=19$, then $11x=\ln 19$, then $x=\frac{\ln 19}{11}$. This is a little hard to think about, but essentially we need to figure out what power of 2.7 produces 19. $2.7^2=7.29$, and $2.7^3=19.683$, so the answer is just less than 3, making $x\cong\frac{3}{11}=.\overline{27}$. This means the exponential function somehow passes through the entire first period of the cosine function, for two more intersections and an answer of 3.

99. I'm working on a 20-foot ladder leaning against a vertical wall and standing on a level floor. Suddenly the foot of the ladder begins to slip away from the wall at a speed of 8 meters per second! At the moment when the foot of the ladder is 16 feet from the wall, how fast is the top of the ladder sliding down the wall?

$x^2+y^2=20^2=400$ and $2x\cdot x'+2y\cdot y'=0$ will be true at all moments. Specifically, $16^2+y^2=400$ becomes $256+y^2=400$, then $y^2=400-256=144$, so that $y=\sqrt{144}=12$. In addition, $2\cdot 16\cdot 8+2\cdot 12\cdot y'=0$ becomes $256=-24y'$, giving $y'=-\frac{256}{24}=-\frac{32}{3}$. This represents the ladder sliding down the wall, making the answer $\frac{32}{3}$.

2016 Team Scramble
Solutions

100. Consider the area bounded by $y = (x+1)^2 + 3$ and $y = 7$. What is the volume passed through by this area when it is rotated around the line $x = 4$?

The area in question is above the parabola, where y ranges from 3 to 7. We can determine x's range by looking at $7 = (x+1)^2 + 3$, which becomes $(x+1)^2 = 7 - 3 = 4$, then $x + 1 = \pm 2$, so that $x = 2 - 1 = 1$ or $x = -2 - 1 = -3$, so x ranges from -3 to 1. If this area is rotated about $x = 4$, we can easily integrate cylindrical areas $A = 2\pi rh$ to get the volume. $2\pi \int_{-3}^{1} (4 - x)\left(7 - ((x+1)^2 + 3)\right)dx = 2\pi \int_{-3}^{1} (4 - x)(3 - 2x - x^2)dx = 2\pi \int_{-3}^{1} (12 - 11x - 2x^2 + x^3)dx = 2\pi \left[12x - \frac{11}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right]_{-3}^{1} = 2\pi \left(12\left(1 - (-3)\right) - \frac{11}{2}(1^2 - (-3)^2) - \frac{2}{3}(1^3 - (-3)^3) + \frac{1}{4}(1^4 - (-3)^4)\right) = 2\pi \left(12 \cdot 4 - \frac{11}{2}(-8) - \frac{2}{3} \cdot 28 + \frac{1}{4}(-80)\right) = 2\pi \left(48 + 44 - \frac{56}{3} - 20\right) = 2\pi \left(72 - \frac{56}{3}\right) = 2\pi \cdot \frac{160}{3} = \frac{320\pi}{3}$